Probability, homework 2, solutions.

Exercise 1. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ a sequence of events in $\mathcal{A}$ which converges to A. Prove that

(i) $A \in \mathcal{A}$;

(ii) $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Solution. As a first step, note that $A_n \to A$ can be rephrased as

$$A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n,$$

where we remind that $\limsup A_n = \cap_{n\geq 1}\cup_{m\geq n} A_m$ and $\liminf A_n = \cup_{n\geq 1}\cap_{m\geq n} A_m$. In particular this implies that $A \in \mathcal{A}$.

We therefore known that $(B_n)_n := (\cup_{m\geq n} A_m)_n$ decreases to A and $(C_n)_n := (\cap_{m\geq n} A_m)_n$ increases to A, so $\mathbb{P}(B_n) \to \mathbb{P}(A)$ and $\mathbb{P}(C_n) \to \mathbb{P}(A)$. As $B_n \subset A_n \subset C_n$, this implies $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbb{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbb{1}_{[2,\infty)}(x)$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Solution We have

$$\begin{aligned}
\mathbb{P}((-1/2,1/2)) &= F(1/2) - F(-1/2) = 1/4 - 0 = 1/4,\\
\mathbb{P}((-1/2,3/2)) &= F(3/2) - F(-1/2) = 3/4 - 0 = 3/4,\\
\mathbb{P}((2/3,5/2)) &= F(5/2) - F(2/3) = 1 - 1/4 = 3/4,\\
\mathbb{P}((3,\infty)) &= F(\infty) - F(3) = 1 - 1 = 0.
\end{aligned}$$

Exercise 3. Let μ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n\geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \mathrm{d}\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n\geq 0}$ does not converge.

Solution Define $f_n(x) = \mathbb{1}_{x\in[f(n),g(n)]}$ where, for $n \in [2^p, 2^{p+1})$, $f(n) = (n - 2^p - 2^{p-1})/p$, $g(n) = f(n) + 1/p$. Clearly $\int f_n = 1/p \to 0$ and for any x we have $f_n(x) = 1$, and 0 i.o.

Exercise 4. Let X be a random variable in $\mathrm{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n\geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_N) \underset{n\to\infty}{\longrightarrow} 0$. Prove that $\mathbb{E}(X\mathbb{1}_{A_n}) \underset{n\to\infty}{\longrightarrow} 0$.

Solution For any $C > 0$ we have

$$|\,\mathbb{E}(X\mathbb{1}_{A_n})| \leq \mathbb{E}(|X|\mathbb{1}_{|X|>C}) + C\mathbb{P}(A_n).$$

Let $\varepsilon > 0$. By monotone convergence, there exists $C > 0$ such that $\mathbb{E}(|X|\mathbb{1}_{|X|>C}) < \varepsilon$. For this C, there exists a n_0 such that for any $n > n_0$ we have $\mathbb{P}(A_n) < \varepsilon/C$. Thus we have proved that for $n > n_0$, $|\,\mathbb{E}(X\mathbb{1}_{A_n})| < 2\varepsilon$, which concludes the proof.

Exercise 5. Let $(d_n)_{n\geq 0}$ be a sequence in $(0,1)$, and $K_0 = [0,1]$. We define iteratively $(K_n)_{n\geq 0}$ in the following way. From K_n, which is the union of closed disjoint intervals, we define K_{n+1} by removing from each interval of K_n an open interval, centered at the middle of the previous one, with length d_n times the length of the previous one. Let $K = \cap_{n\geq 0} K_n$ (K is called a Cantor set).

(a) Prove that K is an uncountable compact set, with empty interior, and whose points are all accumulation points

(b) What is the Lebesgue measure of K?

Solution. (i) Each K_n being closed, so is K. Moreover $K \subset [0, 1]$, so it is compact.

To prove that K is uncountable, consider the following bijection $\varphi : K \to \{0, 1\}^{\mathbb{N}}$: If $x \in K$, then x is either in the left or right interval from K_1, and define $\varphi(x)_0 = 0$ if x is in the left interval, 1 otherwise. Iterations on K_2, etc defines $\varphi(x)$, and φ is easily shown to be a bijection.

To prove that K has empty interior, assume $x, y \in I$ for some interval $I \subset K$. Then for any n we have x, y in the same interval from K_n, i.e. $\varphi(x)_n = \varphi(y)_n$. As φ is a bijection this imposes $x = y$, so I needs to be a point, i.e. K has empty interior.

Finally, for any $x \in K$, the set $\{y \in K : \forall k \leq n, \varphi(y)_k = \varphi(x)_k\}$ is infinite and its points are at distance at most $2^{-(n+1)}$ from x, so x is an accumulation point.

(ii) An easy induction shows that the Lebesgue measure of K_n is $(1-d_0)\dots(1-d_{n-1})$. So

$$\mathrm{Leb}(K) = \lim_{n\to\infty} (1 - d_0) \dots (1 - d_{n-1}).$$

This is 0 if the series of d_n's diverges, a number in $(0, 1)$ otherwise (for this analysis, take the logarithm and Taylor-expand).

Exercise 6. Let X be a nonnegative random variable. Prove that $\mathbb{E}(X) < +\infty$ if and only if $\sum_{n\in\mathbb{N}} \mathbb{P}(X \geq n) < \infty$.

Solution. By monotone convergence we have $\mathbb{E}(X) = \sum_{n\geq 0} \mathbb{E}(X \mathbb{1}_{X\in[n,n+1)})$, so that

$$-1+\sum_{n\geq 0}(n+1)\mathbb{P}(X \in [n, n+1)) = \sum_{n\geq 0} n\mathbb{P}(X \in [n, n+1)) \leq \mathbb{E}(X) \leq \sum_{n\geq 0}(n+1)\mathbb{P}(X \in [n, n+1)).$$

The result follows by noting that $\sum_{n\geq 0}(n+1)\mathbb{P}(X \in [n, n+1)) = \sum_{n\in\mathbb{N}} \mathbb{P}(X \geq n)$.

Exercise 7. *Convergence in measure.* Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. and $(f_n)_{n\geq 1}, f : \Omega \to \mathbb{R}$ measurable (for the Borel σ-field on $\mathbb{R}$). We say that $(f_n)_{n\geq 1}$ converges in measure to f if for any $\varepsilon > 0$ we have

$$\mu(|f_n - f| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0.$$

(i) Show that $\int |f - f_n| \mathrm{d}\mu \to 0$ implies that f_n converges to f in measure. Is the reciprocal true?

(ii) Show that if $f_n \to f$ μ-almost surely, then $f_n \to f$ in measure. Is the reciprocal true?

(iii) Show that if $f_n \to f$ in measure, there exists a subsequence of $(f_n)_{n\geq 1}$ which converges μ-almost surely.

(iv) *(A stronger dominated convergence theorem)* We assume that $f_n \to f$ in measure and $|f_n| \leq g$ for some integrable $g : \Omega \to \mathbb{R}$, for any $n \geq 1$.

  (a) Show that $|f| \leq g$ μ-a.s.

  (b) Deduce that $\int |f_n - f| \mathrm{d}\mu \to 0$.

Solution.

(i) For any $\varepsilon > 0$ we have $\mathbb{1}_{|f-f_n|>\varepsilon} \leq \frac{|f-f_n|}{\varepsilon}$, so that

$$\mu(|f - f_n| > \varepsilon) \leq \frac{1}{\varepsilon} \int_\Omega |f_n - f| \mathrm{d}\mu \to 0.$$

The reciprocal is wrong, as shown by the example $(f_n)_{n\geq 1}$ defined on $([0,1], \mathcal{B}, \mathrm{Leb})$ by $f_n = n\mathbb{1}_{[0,1/n]}$.

(ii) For any $\varepsilon > 0$ we have

$$\cap_{n\geq 1} \cup_{m\geq n} \{|f_n - f| > \varepsilon\} \subset \{f_n \nrightarrow f\}$$

so that $\mu(\cap_{n\geq 1} \cup_{m\geq n} \{|f_m - f| > \varepsilon\}) = 0$. The sequence $(\cup_{m\geq n}\{|f_m - f| > \varepsilon\})_{n\geq 1}$ decreases, so this implies that

$$\lim_{n\to\infty} \mu(\cup_{m\geq n}\{|f_m - f| > \varepsilon\}) = 0,$$

and in particular $\lim_{n\to\infty} \mu(\{|f_n - f| > \varepsilon\}) = 0$.

The reciprocal is wrong, as shown by the example $(f_{n,k})_{n\geq 1, 1\leq k\leq n}$ defined on $([0,1], \mathcal{B}, \mathrm{Leb})$ by $f_{n,k} = \mathbb{1}_{[(k-1)/n, k/n]}$.

(iii) From the hypothesis, for any $k \geq 1$ there exists an index n_k such that $\mu(|f_{n_k} - f| > 1/k) \leq 2^{-k}$. Summability in k easily implies

$$\mu\left(\cap_{m\geq 1} \cup_{k\geq m} \{|f_{n_k} - f| > 1/k\}\right) = 0.$$

As $f_{n_k} \nrightarrow f$ is included in the above set, this concludes the proof.

(iv) (a) For any $\varepsilon > 0$ we have

$$\mu(\{|f| > g + \varepsilon\}) \leq \mu(\{|f| > |f_n| + \varepsilon\}) + \mu(\{|f - f_n| > \varepsilon\})$$

so that $\mu(\{|f| > g + \varepsilon\}) = 0$. Therefore μ-a.s. for any $n \geq 1$ we have $|f| \leq g + 1/n$, so $|f| \leq g$.

(b) We have

$$\int_\Omega |f_n - f| \mathrm{d}\mu = \int_{|f_n - f| < \varepsilon} |f_n - f| \mathrm{d}\mu + \int_{|f_n - f| \geq \varepsilon} |f_n - f| \mathrm{d}\mu \leq \varepsilon + 2\int_{|f_n - f| \geq \varepsilon} |g| \mathrm{d}\mu.$$

As g is integrable, $\int_{|f_n - f|\geq\varepsilon} |g|\mathrm{d}\mu \to 0$ as $n \to \infty$ (e.g. by dominated convergence or uniform continuity of the integral), and the conclusion follows as ε is arbitrary.

Exercise 8. Consider a probability space $(\Omega, \mathcal{A}, \mu)$ and $(A_n)_n$ a sequence in $\mathcal{A}$. Let $f : \Omega \to \mathbb{R}$ be measurable (for the Borel σ-field on $\mathbb{R}$) such that $\int_\Omega |\mathbb{1}_{A_n} - f| \mathrm{d}\mu \to 0$ as $n \to \infty$. Prove that there exists $A \in \mathcal{A}$ such that $f = \mathbb{1}_A$ μ-a.s., i.e. $\mu(f = \mathbb{1}_A) = 1$.

Solution. We first prove that $|f| < 2$ μ-a.s.: $\{|f| > 2\} \subset \{|f - \mathbb{1}_{A_n}| > 1\}$, so that

$$\mu(\{|f| > 2\}) \leq \mu(\{|f - \mathbb{1}_{A_N}| > 1\}) \leq \int_\Omega |\mathbb{1}_{A_n} - f| \mathrm{d}\mu \to 0,$$

where for the second inequality we used $\mu(X > 1) \leq \mathbb{E}(X)$ for any positive random variable. This proves $|f| < 2$ μ-a.s.

We now prove that $f = f^2$ μ-a.s., so that the expected result follows from the choice $A = \{f = 1\}$. We have

$$\begin{aligned}\int_\Omega |f - f^2| \mathrm{d}\mu &\le \int_\Omega |f - \mathbb{1}_{A_n}| \mathrm{d}\mu + \int_\Omega |f^2 - \mathbb{1}_{A_n}| \mathrm{d}\mu \\ &= \int_\Omega |f - \mathbb{1}_{A_n}| \mathrm{d}\mu + \int_\Omega |f - \mathbb{1}_{A_n}| \cdot |f + \mathbb{1}_{A_n}| \mathrm{d}\mu \le 4 \int_\Omega |\mathbb{1}_{A_n} - f| \mathrm{d}\mu \to 0.\end{aligned}$$

where we used that μ-a.s. $|f + \mathbb{1}_{A_n}| \le |f| + 1 \le 3$. This concludes the proof.

Exercise 9. Consider a probability space $(E, \mathcal{A}, \mu)$ and $f_n : E \to \mathbb{R}$ measurable, $n \ge 1$. Assume $f_n \to f$ μ-almost surely. Prove that for any $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ such that $\mu(A) < \varepsilon$ and the convergence $f_n \to f$ is uniform on A^{c}.

Solution. Let

$$E_{k,n} = \cap_{j \ge n} \{|f_j - f| \le 2^{-k}\}.$$

As $f_j \to f$ a.s. , $(E_{k,n})_{n \ge 1}$ is an increasing sequence converging to $E - N$ for some measurable N with $\mu(N) = 0$. For any k let n_k be such that $\mu(E_{k,n_k}) \ge 1 - \frac{\varepsilon}{2^k}$, and let

$$A = N \cup (\cup_{k \ge 1} (E - E_{k,n_k})).$$

Then $\mu(A) \le \varepsilon$ and f_n converges to f uniforly on A^{c}: for any $\omega \in A^{\mathrm{c}}$ we have $x \in E_{k,n_k}$ so that $|f_n(x) - f(x)| \le 2^{-k}$ for all $n \ge n_k$.