

Probability, homework 3, solutions.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and A, B are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Solution. As these events are independent, $\mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$, so either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Exercise 2. Let X be a nonnegative random variable with null expectation. Prove that it is 0 almost surely.

Solution. As X is nonnegative, for any $n \geq 1$ we have $\mathbb{P}(X > n^{-1}) \leq n \mathbb{E}(X) = 0$, so $\mathbb{P}(X \leq n^{-1}) = 1$, hence $\mathbb{P}(\cap_{n \geq 1} \{X \leq n^{-1}\}) = 1$, i.e. $\mathbb{P}(X > 0) = 0$.

Exercise 3. Calculate $\mathbb{E}(X)$ for the following probability measures $\mathbb{P}^X$.

- (i) $\mathbb{P}^X$ has Gaussian density $\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$;
- (ii) $\mathbb{P}^X$ has exponential density $\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$ for some $\lambda > 0$;
- (iii) $\mathbb{P}^X = p\delta_a + q\delta_b$ where $p + q = 1$, $p, q \geq 0$ and $a, b \in \mathbb{R}$;
- (iv) $\mathbb{P}^X$ is the Poisson distribution: $\mathbb{P}^X(\{n\}) = e^{-\lambda} \frac{\lambda^n}{n!}$ for any integer $n \geq 0$, for some $\lambda > 0$.

Solution. For (i), we know that X has the same distribution as $\mu + \sigma Y$ where $Y \sim \mathcal{N}(0, 1)$, so that $\mathbb{E}(X) = \mu$.

For (ii), the relevant calculation is $\mathbb{E}(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx = \int_0^\infty e^{-\lambda x} dx = \lambda^{-1}$, where we used integration by parts.

For (iii) we have $\mathbb{E}(X) = \mathbb{E}(X \mathbb{1}_{X=a}) + \mathbb{E}(X \mathbb{1}_{X=b}) = ap + bq$.

Finally, for (iv) we write (changing the variable $n - 1 = m$)

$$\mathbb{E}(X) = \sum_{n \geq 0} n \mathbb{P}^X(\{n\}) = \sum_{n \geq 0} n e^{-\lambda} \frac{\lambda^n}{n!} = \lambda e^{-\lambda} \sum_{m \geq 0} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} e^\lambda = \lambda.$$

Exercise 4. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Solution. We calculate the cumulative distribution function for any $u \geq 0$

$$\mathbb{P}(X^{-2} < u) = \mathbb{P}(|X| > u^{-1/2}) = 2 \int_{u^{-1/2}}^\infty \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 2 \int_u^\infty \frac{e^{-1/(2v)}}{2\sqrt{2\pi}} v^{-3/2} dv.$$

Differentiating in u we obtain that X^{-2} has density $\frac{e^{-1/(2v)}}{\sqrt{2\pi}} v^{-3/2}$.

Exercise 5. Let X be uniformly distributed on $[0, 1]$ and $\lambda > 0$. Show that $-\lambda^{-1} \log X$ has the same distribution as an exponential random variable with parameter λ .

Solution. We calculate the cumulative distribution function for any $u \geq 0$

$$\mathbb{P}(-\lambda^{-1} \log X > u) = \mathbb{P}(X < \exp(-\lambda u)) = e^{-\lambda u},$$

meaning that $-\lambda^{-1} \log X$ has the same distribution as an exponential random variable with parameter λ .

Exercise 6. A samourai wants to create a triangle with a (rigid) spaghetti. With his saber, he cuts this spaghetti on two places, chosen uniformly and independently along this traditional pasta. What is the probability that he can create a triangle

with sides these three pieces of spaghetti?

Solution. Without loss of generality we assume that the spaghetti length is 1. Let X and Y be the coordinates of the cuts along the spaghetti: The distribution of (X, Y) is the product measure $\mu \times \mu$ where μ is the Lebesgue measure on $[0, 1]$.

The samourai can create a triangle if and only if the side lengths satisfy the triangle inequalities (denote A this event). By symmetry $\mathbb{P}(A)$ coincides with

$$\begin{aligned} & 2\mathbb{P}(A \cap \{X < Y\}) \\ &= 2\mathbb{P}(X < Y, X < (Y - X) + (1 - Y), Y - X < X + (1 - Y), 1 - Y < X + (Y - X)) \\ &= 2\mathbb{P}(X < Y, X < 1/2, Y - 1/2 < X, 1/2 < Y) \\ &= 2 \iint_{x < y, x < 1/2 < y, y - 1/2 < x} dx dy = 2 \cdot \frac{1}{8} = \frac{1}{4}. \end{aligned}$$

Exercise 7. Assume that $X_1, X_2, \dots$ are independent random variables uniformly distributed on $[0, 1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \leq i \leq n\}$. Prove that it converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$\mathbb{E}\left(f(Y^{(n)})\right) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^+} f(u) e^{-u} du.$$

Solution. We have

$$\begin{aligned} \mathbb{P}(n \inf(X_1, \dots, X_n) > x) &= \prod_{k=1}^n \mathbb{P}(X_k > x/n) \\ &= (1 - x/n)^n \\ &\rightarrow \exp(-x) \\ &= \int_x^\infty e^{-u} du, \end{aligned}$$

and the desired equation follows by integration over x . In other words, $n \inf(X_1, \dots, X_n)$ converges in distribution to an exponential random variable with parameter 1.

Exercise 8. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. For any $y \in \mathbb{R}$, let $N(y) \in \mathbb{R} \cup \{\infty\}$ be the number of solutions to $f(x) = y$. Prove that N is measurable.

Solution. The equation $f(x) = y$ has a solution in $[\frac{k}{2^n}, \frac{k+1}{2^n})$ if and only if $y \in f([\frac{k}{2^n}, \frac{k+1}{2^n}))$. By continuity $f([\frac{k}{2^n}, \frac{k+1}{2^n}))$ is an interval, hence it is measurable. This implies that the function N_n defined by

$$N_n(y) = \sum_{k=0}^{2^n-1} \mathbb{1}_{y \in f([\frac{k}{2^n}, \frac{k+1}{2^n}))} + \mathbb{1}_{y=f(1)}$$

is measurable. Moreover, N_n converges pointwise to N . Hence N is measurable.

Exercise 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be additive ($f(a + b) = f(a) + f(b)$ for all a, b) and measurable. Prove that it is linear.

What if f is not assumed to be measurable? What about $f : \mathbb{R}^d \rightarrow \mathbb{R}$?

Solution. Assume $f(1) = 1$ w.l.o.g. If f is additive, $bf(1/b) = f(1) = 1$ for any positive integer b , which then also implies $f(a/b) = af(1/b) = a/b$ for all non-zero integers a, b .

On the one hand, this implies that if f is additive and non-linear, there exists $x \notin \mathbb{Q}$ such that $f(x) \neq x$. For any $\varepsilon > 0$ there exists integers n, m arbitrary large such that $|nx - m| < \varepsilon$. We naturally have

$$f(nx - m) = nf(x) - m = nx - m + n(f(x) - x),$$

so that we have proved that for any $\varepsilon, M > 0$ there exists a real $u = nx - m \in [-\varepsilon, \varepsilon]$ such that $|f(u)| > M$: if f is additive and non-linear, it is unbounded in any neighborhood of 0.

On the other hand, if f is additive and measurable, it needs to be bounded in some neighborhood of 0. Indeed, let $A_n = \{x : |f(x)| \leq n\}$. By assumption A_n is measurable. It is also increasing to $\mathbb{R}$ so $\text{Leb}(A_{n_0}) > 0$ for some n_0 . By the lemma below, $A_{n_0} - A_{n_0}$ contains an interval $[-\varepsilon, \varepsilon]$, and any $x \in [-\varepsilon, \varepsilon]$ therefore satisfies

$$|f(x)| = f(a - b) = |f(a) - f(b)| \leq 2n_0$$

for some $a, b \in A_{n_0}$: f is bounded in $[-\varepsilon, \varepsilon]$. This concludes the proof.

Lemma. If A is measurable and $\text{Leb}(A) > 0$, $A - A := \{a - b, (a, b) \in A^2\}$ contains an interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$. (This is Steinhaus' theorem)

Proof of the Lemma. W.l.o.g. A is compact. Thus there exists an open set U such that $A \subset U$ and $\text{Leb}(U) < 2\text{Leb}(A)$. Let $\varepsilon > 0$ be such that $A + [-\varepsilon, \varepsilon] \subset U$. Then any $x \in [-\varepsilon, \varepsilon]$ is also in $A - A$: If $(x + A) \cap A = \emptyset$, then we would have $\text{Leb}(U) \geq \text{Leb}((x + A) \cup A) = 2\text{Leb}(A)$, a contradiction.

Note that the above proof works for additive measurable functions on $\mathbb{R}^d$ up to cosmetic changes.

Moreover, to construct a counterexample when f is non-measurable, consider a (uncountable) basis A of $\mathbb{R}$ as a $\mathbb{Q}$ -vector space. Say $1 \in A$ and $\sqrt{2} \in A$. Let $f(1) = 1$, $f(\sqrt{2}) = \sqrt{3}$, and f defined arbitrarily on other points in A . Then f is extended naturally by $\mathbb{Q}$ -linearity to $\text{span}(A) = \mathbb{R}$. It is then additive, but easily seen not to be continuous.