

Probability, homework 4, solutions.

Exercise 1. Let n and m be random numbers chosen independently and uniformly on $\llbracket 1, N \rrbracket$. What are Ω , $\mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on N) ? Prove that $\mathbb{P}(n \wedge m = 1) \xrightarrow{N \rightarrow \infty} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p \in \mathcal{P}} (1 - p^{-2})^{-1} = \sum_{n \geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don't have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n \wedge m = 1$ means that their greatest common divisor is 1.

Solution. We can choose $\Omega = \llbracket 1, N \rrbracket^2$, $\mathcal{A} = \{A : A \subset \Omega\}$ and define $\mathbb{P}$ through $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$. We denote the random variable $(n, m) = (n(\omega), m(\omega))$. Let $A_p = \{p \mid n \text{ and } p \mid m\}$ and $f_N = \#\{p \leq N, p \in \mathcal{P}\}$. Then, by inclusion-exclusion,

$$\begin{aligned} \mathbb{P}(n \wedge m = 1) &= \mathbb{P}(\cap_{p \leq N} A_p^c) = 1 - \mathbb{P}(\cup_{p \leq N} A_p) \\ &= 1 - \sum_{p_1 \leq N} \mathbb{P}(A_{p_1}) + \sum_{p_1 < p_2 \leq N} \mathbb{P}(A_{p_1} \cap A_{p_2}) + \dots + (-1)^{f_N} \mathbb{P}(A_{p_1} \cap A_{p_2} \cap \dots \cap A_{p_{f_N}}) \\ &= 1 - \sum_{p_1 \leq N} \left(\frac{\lfloor N/p_1 \rfloor}{N} \right)^2 + \sum_{p_1 < p_2 \leq N} \left(\frac{\lfloor N/(p_1 p_2) \rfloor}{N} \right)^2 - \dots \end{aligned}$$

When replacing the integral parts by fractions, we obtain

$$1 - \sum_{p_1 \leq N} p_1^{-2} + \sum_{p_1 < p_2 \leq N} (p_1 p_2)^{-2} - \dots + (-1)^{f_N} (p_1 \dots p_{f_N})^{-2} = (1 - p_1^{-2}) \dots (1 - p_{f_N}^{-2}),$$

which converges to the expected result as $N \rightarrow \infty$. Hence we now just need to bound the error when replacing integral parts with fractions. To bound this error we use $|a^2 - \lfloor a \rfloor^2| \leq (a - \lfloor a \rfloor) \cdot 2a$ for any $a > 0$ so that

$$\left| \left(\frac{\lfloor N/(p_1 p_2 \dots p_k) \rfloor}{N} \right)^2 - \left(\frac{N/(p_1 p_2 \dots p_k)}{N} \right)^2 \right| \leq \frac{2}{N p_1 p_2 \dots p_k}$$

and the global error is at most

$$N^{-1} \left(\sum_{p_1 \leq N} p_1^{-1} + \sum_{p_1 < p_2 \leq N} (p_1 p_2)^{-1} + \dots + (p_1 \dots p_{f_N})^{-1} \right) = N^{-1} (1 + p_1^{-1}) \dots (1 + p_{f_N}^{-1}) \rightarrow 0,$$

where the last convergence relies on $\log[(1 + p_1^{-1}) \dots (1 + p_{f_N}^{-1})] \sim \sum_{p \leq N} 1/p \sim \log \log N$.

Exercise 2. Let X be a random variable with density $f_X(x) = (1 - |x|)\mathbf{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1 - \cos u)}{u^2}.$$

Solution. Let's calculate:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} e^{iux} f_X(x) dx &= \operatorname{Re} \int_0^1 (1 - x) e^{iux} dx \\ &= \operatorname{Re} \int_0^1 e^{iux} dx - \operatorname{Re} \frac{e^{iu}}{iu} + \operatorname{Re} \int_0^1 \frac{e^{iux}}{iu} dx \\ &= \operatorname{Re} \frac{e^{iu} - 1}{iu} - \operatorname{Re} \frac{e^{iu}}{iu} + \operatorname{Re} \frac{e^{iu} - 1}{(iu)^2} \\ &= -\operatorname{Re} \frac{e^{iu} - 1}{u^2}, \end{aligned}$$

where we used integration by parts in the second equality. This concludes the proof.

Exercise 3.

- (1) Prove that $\hat{\mu}$ is real-valued if and only if μ is symmetric, i.e. $\mu(A) = \mu(-A)$ for any Borel set A
- (2) If X and Y are i.i.d., prove that $X - Y$ has a symmetric distribution.

Solution.

- (1) If $\mu(A) = \mu(-A)$ for any Borel set A , then

$$\overline{\hat{\mu}(u)} = \int_{-\infty}^{\infty} e^{-iu x} \mu(dx) = \int_{-\infty}^{\infty} e^{iu x} \mu(-dx) = \int_{-\infty}^{\infty} e^{iu x} \mu(dx) = \hat{\mu}(u),$$

so μ is real-valued.

Conversely, from the previous calculation, if $\hat{\mu}$ is real-valued, then for any $u \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} e^{iu x} (\mu(dx) - \mu(-dx)) = 0.$$

By characterization through Fourier transform, this implies that $\mu(dx) - \mu(-dx) = 0$ (as a signed measure), hence the claim.

- (2) From the previous question we just need to prove that the characteristic function is real-valued, which is true:

$$\mathbb{E}(e^{iu(X-Y)}) = \mathbb{E}(e^{iuX}) \mathbb{E}(e^{-iuY}) = \mathbb{E}(e^{iuX}) \mathbb{E}(e^{-iuX}) = \mathbb{E}(e^{iuX}) \overline{\mathbb{E}(e^{iuX})} \in \mathbb{R}.$$

Exercise 4. Let X_λ be a real random variable, with Poisson distribution with parameter λ . Calculate the characteristic function of X_λ . Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \rightarrow \infty$.

Solution. Let's simply calculate:

$$\mathbb{E}(e^{iuX_\lambda}) = \sum_{k \geq 0} \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} = e^{\lambda(e^{iu} - 1)}.$$

The desired weak convergence now follows from the choice $u = v/\sqrt{\lambda}$ and Taylor expansion in λ for fixed v .

Exercise 5. Assume that the sequence of random variables $(X_n)_{n \geq 1}$ satisfies $\mathbb{E} X_n \rightarrow 1$ and $\mathbb{E} X_n^2 \rightarrow 1$. Prove that $(X_n)_{n \geq 1}$ converges in distribution. What is the limit?

Solution. We will prove that for smooth compactly supported f we have $\mathbb{E} f(X_n) \rightarrow \mathbb{E} f(1)$, which is sufficient to characterize the weak convergence of X_n to 1. We have $|f(X_n) - f(1)| \leq \|f'\|_\infty \cdot |X_n - 1|$, so that by taking expectation first and using the Cauchy-Schwarz inequality after we obtain

$$\mathbb{E} |f(X_n) - f(1)| \leq \|f'\|_\infty \cdot \mathbb{E} |X_n - 1| \leq \|f'\|_\infty \cdot (\mathbb{E} |X_n - 1|^2)^{1/2}.$$

By expansion, and using our assumptions, we have $\mathbb{E} |X_n - 1|^2 \rightarrow 0$, which concludes the proof.

Exercise 6. Let $(X_n)_{n \geq 1}$, $(Y_n)_{n \geq 1}$ be real random variables, with X_n and Y_n independent for any $n \geq 1$, and assume that X_n converges in distribution to X and Y_n to Y . Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

Solution. The proof is easy with characteristic functions, using successively independence and weak convergence: for any $\lambda \in \mathbb{R}$ we have

$$\mathbb{E}(e^{i\lambda(X_n+Y_n)}) = \mathbb{E}(e^{i\lambda X_n}) \mathbb{E}(e^{i\lambda Y_n}) \xrightarrow{n \rightarrow \infty} \mathbb{E}(e^{i\lambda X}) \mathbb{E}(e^{i\lambda Y}) = \mathbb{E}(e^{i\lambda(X+Y)}).$$

Exercise 7. Let X, Y be independent and assume that for some constant α we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that X and Y are both constant random variables.

Solution. Assume the contrary, i.e. e.g. there exists $[a, b]$ and $[c, d]$ with $b < c$ such that $\mathbb{P}(X \in [a, b]) > 0$, $\mathbb{P}(X \in [c, d]) > 0$.

Let $[x, y]$ be such that $y - x < c - b$ and $\mathbb{P}(Y \in [x, y]) > 0$.

Then we have

$$\mathbb{P}(X + Y > c + x) \geq \mathbb{P}(X > c, Y > x) > 0$$

and

$$\mathbb{P}(X + Y < b + y) \geq \mathbb{P}(X < b, Y < y) > 0.$$

The two equations above together with $c + x > b + y$ contradict the fact that $X + Y$ has distribution a Dirac mass.

Exercise 8. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing measurable functions. Let μ be a probability measure on $\mathbb{R}$ and assume $f, g, fg \in L^1(\mu)$. prove that

$$\int fg d\mu \geq \int f d\mu \cdot \int g d\mu.$$

Solution. We have $\iint (f(x) - f(y)) \cdot (g(x) - g(y)) d\mu(x) d\mu(y) \geq 0$. Then the result follows from an expansion and Fubini's theorem.

Exercise 9. Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution and let $M_n = \max(X_1, \dots, X_n)$. Prove that $(nM_n^{-1})_{n \geq 1}$ converges in distribution and identify the limit.

Solution. For any fixed $x > 0$ we have

$$\begin{aligned}
\mathbb{P}(nM_n^{-1} > x) &= \mathbb{P}(M_n < n/x, M_n > 0) \\
&= \mathbb{P}(M_n < n/x) + O(\mathbb{P}(M_n < 0)) \\
&= \prod_{k=1}^n \mathbb{P}(X_k < n/x) + O(2^{-n}) \\
&= \left(1 - \int_{n/x}^{\infty} \frac{du}{\pi(1+u^2)}\right)^n + O(2^{-n}) \\
&= \exp\left(n \log\left(1 - \frac{1}{\pi n/x} + O(1/n^2)\right)\right) + O(2^{-n}) \\
&\rightarrow \exp(-x/\pi) \\
&= \int_x^{\infty} \pi^{-1} e^{-u/\pi} du,
\end{aligned}$$

where we used independence for the second equality. Therefore nM_n^{-1} converges in distribution to an exponential random variable with parameter π^{-1} .

Exercise 10. Let $(X_i)_{i \geq 1}$ be a sequence of independent random variables, with X_i uniform on $[-i, i]$. Let $S_n = X_1 + \dots + X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

Solution. Let's calculate the characteristic function (Y_k 's denote independent uniform random variables on $[-1, 1]$):

$$\begin{aligned}
\mathbb{E}(\exp(iuS_n n^{-3/2})) &= \prod_{k=1}^n \mathbb{E}(\exp(iun^{-3/2}X_k)) = \prod_{k=1}^n \mathbb{E}(\exp(iun^{-3/2}kY_k)) = \prod_{k=1}^n \frac{\sin(un^{-3/2}k)}{un^{-3/2}k} \\
&= \prod_{k=1}^n \left(1 - \frac{(un^{-3/2}k)^3}{6un^{-3/2}k} + O((n^{-3/2}k)^4)\right) = \prod_{k=1}^n \exp\left(-\frac{(un^{-3/2}k)^2}{6} + O((n^{-3/2}k)^4)\right)
\end{aligned}$$

As $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6 \sim n^3/3$, we obtain

$$\mathbb{E}(\exp(iuS_n n^{-3/2})) \rightarrow e^{-\frac{u^2}{2 \cdot 3^2}},$$

in other words $S_n n^{-3/2}$ converges in distribution to a Gaussian random variable with mean zero and variance $1/3^2$.

Exercise 11. Find a probability distribution μ of a $\mathbb{Z}$ -valued random variable which is symmetric ($\mu(\{i\}) = \mu(\{-i\})$) for any $i \in \mathbb{Z}$, not integrable, but such that its characteristic function is differentiable at 0.

Solution. One example is $\mu(\{n\}) = \mu(\{-n\}) = \frac{c}{n^2 \log n}$, $n \geq 2$, 0 otherwise, where c is a normalizing constant. Clearly $\int |x| d\mu(x) = \infty$, but

$$\hat{\mu}(u) = c' \sum_{k \geq 2} \frac{\cos(ku)}{k^2 \log k}$$

is differentiable at 0 because

$$\frac{\hat{\mu}(\varepsilon) - \hat{\mu}(0)}{\varepsilon} = \frac{c''}{\varepsilon} \sum_{k=2}^{(\varepsilon \sqrt{\log \varepsilon^{-1}})^{-1}} \frac{1 - \cos(k\varepsilon)}{k^2 \log k} + O\left(\frac{1}{\sqrt{\log \varepsilon^{-1}}}\right)$$

From $|1 - \cos x| \leq Cx^2$ one gets the result that $\hat{\mu}$ is differentiable at 0, with differential 0.

Exercise 12 Let X, Y be i.i.d., with characteristic functions denoted φ_X, φ_Y , and suppose $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = 1$. Assume also that $X+Y$ and $X-Y$ are independent.

(1) Prove that

$$\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)$$

(2) Prove that X is a standard Gaussian random variable.

Solution.

(1) Let's calculate:

$$\begin{aligned} \varphi_X(2u) &= \mathbb{E}(e^{iu(X+Y+X-Y)}) = \mathbb{E}(e^{iu(X+Y)}) \mathbb{E}(e^{iu(X-Y)}) \\ &= \mathbb{E}(e^{iuX}) \mathbb{E}(e^{iuY}) \mathbb{E}(e^{iuX}) \mathbb{E}(e^{-iuY}) = (\mathbb{E}(e^{iuX}))^3 \mathbb{E}(e^{-iuX}), \end{aligned}$$

which concludes the proof of the first question.

(2) This is a truly hard question. Let $\varphi = \varphi_X$. By induction on n , one first obtains (from the previous question applied to u and $-u$) that

$$\frac{\varphi(u)}{\varphi(-u)} = \left(\frac{\varphi(u2^{-n})}{\varphi(-u2^{-n})} \right)^{2^n}$$

for any $n \geq 0$. This actually holds only for $|u| < \varepsilon$ where ε is small enough so that none denominator above vanishes (possible by dominated convergence, think about it!).

Now let $n \rightarrow \infty$ in the previous equation. As $\mathbb{E}(X) = 0$, and $\mathbb{E}(X^2) < \infty$, a Taylor expansion of the exponential shows that the right hand side converges to 1.

We have therefore proved that $\varphi(u) = \varphi(-u)$ for $|u| < \varepsilon$. This information injected in the result of Section 1 gives $\varphi(2u) = \varphi(u)^4$. Now an iteration of this relation provides

$$\varphi(u) = \varphi(u/2^n)^{4^n}.$$

By Taylor expansion again, the above right hand side converges, as $n \rightarrow \infty$, to $e^{-u^2/2}$. This holds in a neighborhood of 0 first, but then this is extended to $\mathbb{R}$. because of (1). This concludes the proof.