Probability, homework 6, solutions.

Exercise 1. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2} = \frac{1}{4} \text{ a.s.}$$

Solution. Let $A = \{\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = 1\}$ and $B = \{\lim_{n\to\infty} \frac{X_1^2+\cdots+X_n^2}{n} = 4\}$. The X_i's and X_i^2's are in L^1 and $\mathbb{E}(X_i) = 1$, $\mathbb{E}(X_i^2) = 3 + \mathbb{E}(X_i)^2 = 4$, so by the law of large numbers $\mathbb{P}(A) = \mathbb{P}(B) = 1$, so that $\mathbb{P}(A \cap B) = 1$. As $A \cap B \subset \{\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2} = \frac{1}{4}\}$, the result follows.

Exercise 2. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1+\cdots+x_n}{n}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Solution. Let $U_1, U_2, \ldots$ be independent random variables, uniform on $[0,1]$. By the law of large numbers, $(U_1+\cdots+U_n)/n \to 1/2$ almost surely, hence in distribution. This implies that for any continuous function on $[0,1]$ (and therefore bounded),

$$\mathbb{E}\, f\left(\frac{U_1+\cdots+U_n}{n}\right) \to f\left(\frac{1}{2}\right),$$

which is equivalent to showinf that the desired asymptotics is $f\left(\frac{1}{2}\right)$.

Exercise 3. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^{n} X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1-x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Solution.

(a) We have $\mathbb{P}(B^{(n,x)} = k) = \binom{n}{k} x^k (1-x)^{n-k}$, so that

$$\mathbb{E}(f(S_n(x))) = \sum_{k=0}^{n} \mathbb{P}(B^{(n,x)} = k) \cdot f\left(\frac{k}{n}\right) = B_n(x).$$

(b) Let $\varepsilon > 0$. As f is continuous on the compact $[0,1]$, it is uniformly continuous, i.e. there exists $\delta > 0$ such that $|f(a) - f(b)| < \varepsilon$ for any $a, b \in [0,1]$ satisfying $|a-b| < \delta$. Then

$$\begin{aligned}
|B_n(x) - f(x)| &= |\,\mathbb{E}(f(S_n(x)) - f(x))| \\
&\leq |\,\mathbb{E}[(f(S_n(x)) - f(x))\mathbb{1}_{|S_n(x)-x|<\delta}]| + |\,\mathbb{E}[(f(S_n(x)) - x)\mathbb{1}_{|S_n(x)-f(x)|\geq\delta}]| \\
&\leq \varepsilon + 2\|f\|_\infty \,\mathbb{P}(|S_n(x) - x| > \delta) \leq \varepsilon + 2\|f\|_\infty \,\delta^{-2}\,\mathbb{E}[(S_n(x)-x)^2].
\end{aligned}$$

We have $\mathbb{E}[(S_n(x)-x)^2] = n^{-2}\mathrm{Var}(\sum X_i) = n^{-2}\sum \mathrm{Var}(X_i) = n^{-1}\mathrm{Var}(X_1)$, so that for $n > 2\|f\|_\infty \frac{\varepsilon}{\delta^2} \max_{0\leq x\leq 1} \mathrm{Var}(X_1)$ we obtain $|B_n(x) - f(x)| \leq 2\varepsilon$, which concludes the proof.

Exercise 4. Calculate

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}.$$

Solution. Let X_n be a Poisson random variable with parameter n. Then the sum of interest can exactly be interpreted as

$$\mathbb{P}(X_n \leq n) = \mathbb{P}(\frac{X_n - n}{\sqrt{n}} \leq 0).$$

Then, as $n \to \infty$, to $\mathbb{P}(\mathcal{N}(0,1) \leq 0) = \frac{1}{2}$ from the central limit theorem for X_n (which can be proved directly by characteristic functions).

Exercise 5. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d random variables, with uniform distribution on $[0,1]$. Let $Y_n = (X_n)^n$.

(1) Calculate the distribution of Y_n.
(2) Show that $(Y_n)_{n\geq 0}$ converges to 0 in probability.
(3) Show that $(Y_n)_{n\geq 0}$ converges in L^1.
(4) Show that almost surely $(Y_n)_{n\geq 0}$ does not converge.

Solution. (1) We just calculate the cumulative distribution function: For any $x \in [0,1]$ we have $\mathbb{P}(Y_n \leq x) = \mathbb{P}(X_n \leq x^{1/n}) = x^{1/n}$.

(2) From the previous calculation, $\mathbb{P}(|Y_n| > \varepsilon) = 1 - \varepsilon^{1/n} \to 0$ as $n \to \infty$, so Y_n converges in probability to 0.

(3) We have $\mathbb{E}[|Y_n|] = \int_0^1 \mathbb{P}(Y_n > x)$, and from the previous question the integrand converges pointwise to 0. So by dominated convergence $\mathbb{E}[|Y_n|] \to 0$ so $Y_n \to 0$ in L^1.

(4) As Y_n convergest to 0 in probability, if it converges a.s. then it can only be to 0. But $\sum_{n\geq 1}\mathbb{P}(|Y_n| > 1/2)$ diverges from the calculation in (2), and the Y_n's independent, so that a.s. $Y_n > 1/2$ infinitely often. Hence ot does not converge a.s.

Long problem. The goal is to prove the Erdős-Kac theorem: if $w(m)$ denotes the number of distinct prime factors of m and k is a random variable uniformly distributed on $[\![1,n]\!]$, then the following convergence in distribution holds:

$$\frac{w(k) - \log\log n}{\sqrt{\log\log n}} \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0,1).$$

(i) Prove that if $(X_n)_{n\geq 1}$ converges in distribution to $\mathcal{N}(0,1)$ and $\sup_{n\geq 1}\mathbb{E}[X_n^{2k}] < \infty$ for any $k \in \mathbb{N}$, then $\lim_{n\to\infty}\mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0,1)^k]$ for any $k \in \mathbb{N}$.
(ii) Prove that for any $x \in \mathbb{R}$ and $d \geq 1$ we have

$$\left| e^{\mathrm{i}x} - \sum_{\ell=0}^{d}\frac{(\mathrm{i}x)^\ell}{\ell!}\right| \leq \frac{|x|^{d+1}}{(d+1)!}.$$

(iii) Assume that $\lim_{n\to\infty}\mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0,1)^k]$ for any $k \in \mathbb{N}$. Prove that X_n converges in distribution to X.
(iv) Let $w_y(m)$ be the number of prime factors of m which are smaller than y. Let $(B_p)_{p \text{ prime}}$ be independent random variables such that $\mathbb{P}(B_p = 1) = 1 - \mathbb{P}(B_p = 0) = \frac{1}{p}$, $W_y = \sum_{p\leq y} B_p$, $\mu_y = \sum_{p\leq y}\frac{1}{p}$, $\sigma_y^2 = \sum_{p\leq y}(\frac{1}{p} - \frac{1}{p^2})$.

Prove that if $y = n^{\mathrm{o}(1)}$, then for any $d \in \mathbb{N}$ we have

$$\mathbb{E}\left[\left(\frac{w_y(k)-\mu_y}{\sigma_y}\right)^d\right] - \mathbb{E}\left[\left(\frac{W_y-\mu_y}{\sigma_y}\right)^d\right] \to 0$$

as $n \to \infty$.

Solution.

(i) For $M > 0$, let $f_M(x) = x^k$ if $|x| < M$, M otherwise. Then

$$\begin{aligned}
&|\,\mathbb{E}[X_n^k - \mathscr{N}(0,1)^k]| \\
\leq &|\,\mathbb{E}[f_M(X_n) - f_M(\mathscr{N}(0,1))]| + |\,\mathbb{E}[|X_n^k|\mathbb{1}_{|X_n|>M}]| + |\,\mathbb{E}[|\mathscr{N}(0,1)^k|\mathbb{1}_{|\mathscr{N}(0,1)|>M}]| \\
\leq &|\,\mathbb{E}[f_M(X_n) - f_M(\mathscr{N}(0,1))]| + |\,\mathbb{E}[|X_n^{2k}/M^k]| + |\,\mathbb{E}[|\mathscr{N}(0,1)^{2k}/M^k]| \\
\leq &|\,\mathbb{E}[f_M(X_n) - f_M(\mathscr{N}(0,1))]| + C_k/M^k.
\end{aligned}$$

For fixed $\varepsilon > 0$, one can find $M > 0$ such that $C_k/M^k < \varepsilon$. For this M, by weak convergence the expectation above converges to 0. Hence for large enough n we have $|\,\mathbb{E}[X_n^k - \mathscr{N}(0,1)^k]| \leq 2\varepsilon$.

(ii) This is an easy exercise from the Taylor formula with integral remainder.

(iii) Let $u \in \mathbb{R}$ be fixed. One easily proves that $\mathbb{E}[|\mathscr{N}(0,1)^{2\ell}|u^{2\ell}] \ll (2\ell)!$. Hence by the previous question for any $\varepsilon > 0$ we can find $d > 0$ such that

$$|\,\mathbb{E}[e^{\mathrm{i}uX_n} - e^{\mathrm{i}u\mathscr{N}(0,1)}]| \leq \sum_{\ell=0}^{d} |\,\mathbb{E}[X_n^\ell - \mathscr{N}(0,1)^\ell]| + \varepsilon.$$

Convergence of the moments then gives $|\,\mathbb{E}[e^{\mathrm{i}uX_n} - e^{\mathrm{i}u\mathscr{N}(0,1)}]| \leq 2\varepsilon$ for large enough n, which concludes the proof.

(iv) Note that, by binomial expansion,

$$\mathbb{E}[(w_y(k)-\mu_y)^d] - \mathbb{E}[(W_y-\mu_y)^d] = \sum_{j=0}^{d} \binom{d}{j} (-\mu_y)^{d-j} (\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j]),$$

thus we only need to prove that for any fixed j,

$$\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j] = \mathrm{o}(\mu_y^{j-\frac{d}{2}}).$$

We have

$$\begin{aligned}
\mathbb{E}[(w_y(k))^j] &= \sum_{p_1,\dots,p_j \leq y} \mathbb{P}(p_1 \mid k, \dots, p_j \mid k) = \sum_{p_1,\dots,p_j \leq y} \frac{\lfloor n/\mathrm{lcm}(p_1,\dots,p_j)\rfloor}{n} \\
&= \sum_{p_1,\dots,p_j \leq y} \frac{1}{\mathrm{lcm}(p_1,\dots,p_j)} + \mathrm{O}(\pi(y)^j/n) = \mathbb{E}[(W_y)^j] + \mathrm{O}(y^j/n) = \mathbb{E}[(W_y)^j] + \mathrm{o}(\mu_y^{j-\frac{d}{2}}),
\end{aligned}$$

concluding the proof.

(v) We now choose $y = n^{1/\log\log\log n}$. From the previous questions, $\frac{w_y(k)-\mu_y}{\sigma_y}$ converges to a standard Gaussian. Any $k \leq n$ has at most $\log n/\log y = \log\log\log n$ prime factors in $[y, n]$, we have $w_y(k) = w(k) + \mathrm{O}(\log\log\log n)$. Moreover a calculation gives $\mu_y = \log\log y + \mathrm{O}(1) = \log\log n + \mathrm{O}(\log\log\log\log n)$ and $\sigma_y^2 \sim \log\log n$, so that $\frac{w(k)-\mu_y}{\sigma_y}$ also converges in distribution to a standard Gaussian. Injecting the estimates on μ_y and σ_y concludes the proof.