

Probability, homework 7, due October 28.

Exercise 1. Read carefully Section 3.8 in Varadhan's *Probability Theory*. Explain the proof of Theorem 3.21 or Lemma 3.22 to a friend, on the board.

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for $X_1, X_2, \dots$ i.i.d. standard Gaussian random variables, denoting $S_n = X_1 + \dots + X_n$, we have

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right) = 1 \quad (0.1)$$

(1) Prove that

$$\mathbb{P}(X_1 > \lambda) \underset{\lambda \rightarrow \infty}{\sim} \frac{1}{\lambda \sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}.$$

In the following questions we denote $f(n) = \sqrt{2n \log \log n}$, $\lambda > 1$, $c, \alpha > 0$, $A_k = \{S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^k)\}$, $C_k = \{S_{\lfloor \lambda^{k+1} \rfloor} - S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^{k+1} - \lambda^k)\}$ and $D_k = \{\sup_{n \in [\lambda^k, \lambda^{k+1}]} \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \geq \alpha\}$.

(2) Prove that for any $c > 1$ we have $\sum_{k \geq 1} \mathbb{P}(A_k) < \infty$ and

$$\limsup_{k \rightarrow \infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \leq 1 \text{ a.s.}$$

(3) Prove that for any $c < 1$ we have $\sum_{k \geq 1} \mathbb{P}(C_k) = \infty$ and

$$\mathbb{P}(C_k \text{ i.o.}) = 1.$$

(4) Let $\varepsilon > 0$ and choose $c = 1 - \varepsilon/10$. Prove that almost surely the following inequality holds for infinitely many k :

$$\frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq c \frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} - (1 + \varepsilon) \frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) By choosing a large enough λ in the previous inequality, prove that almost surely

$$\limsup_{n \rightarrow \infty} \frac{S_n}{f(n)} \geq 1.$$

(6) Prove that for any $n \in [\lambda^k, \lambda^{k+1}]$ and $S_n > 0$ we have

$$\frac{S_n}{f(n)} \leq \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)} + \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)}.$$

(7) Prove that

$$\mathbb{P}(D_k) \underset{k \rightarrow \infty}{\sim} 2\mathbb{P} \left(X_1 \geq \frac{\alpha f(\lambda^k)}{\sqrt{\lambda^{k+1} - \lambda^k}} \right) \underset{k \rightarrow \infty}{\sim} \frac{\text{cst}}{\sqrt{\log k}} \left(\frac{1}{k} \right)^{\frac{\alpha^2}{\lambda-1}}.$$

(8) Prove that for $\alpha^2 > \lambda - 1$, almost surely

$$\limsup_{n \rightarrow \infty} \frac{S_n}{f(n)} \leq \limsup_{n \rightarrow \infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} + \alpha.$$

(9) By choosing appropriate λ and α , prove that almost surely

$$\limsup_{n \rightarrow \infty} \frac{S_n}{f(n)} \leq 1.$$

(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?