Random matrix ensembles generated by Lax matrices

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Outline

- Standard random matrix ensembles and dynamical systems
- Systems and ensembles with intermediate statistics
- Quantization of a pseudo-integrable map
- Classical integrable systems and related random matrix ensembles
- Calogero-Moser ensembles
- Ruijsenaars-Schneider ensembles
- Conclusion

Main ensembles

- **Poisson ensemble of diagonal matrices**

$$\mathbf{M_{ij}} = \mathbf{p_j}\delta_{\mathbf{ij}} \text{ with } \mathbf{p_j} = \text{i.i.d. random variables}$$

- **Standard Random Matrix Ensembles**

$\mathcal{M} \equiv \mathbf{M_{ij}} =$ real symmetric, complex Hermitian, or quaternion matrices

Measure: $\mathrm{e}^{-a\,\mathrm{Tr}\,\mathcal{M}\mathcal{M}^\dagger}$ is invariant over conjugation

$$\mathcal{M} \longrightarrow U\,\mathcal{M}\,U^{-1}$$

of a group of orthogonal, unitary, or symplectic matrices.

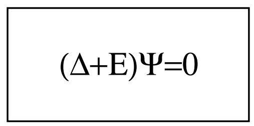
Joint distribution of eigenvalues

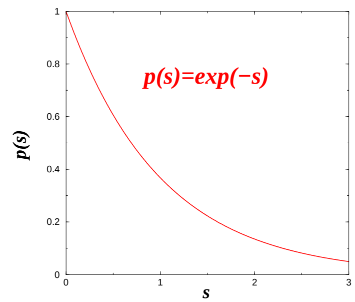
$$P(\lambda) \sim \prod_{j<k} |\lambda_k - \lambda_j|^{\beta} \mathrm{e}^{-\sum_s V(\lambda_s)}$$

$\beta = 1,\ 2,\ 4$ for GOE, GUE, and GSE

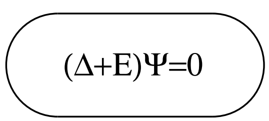
Well accepted conjectures

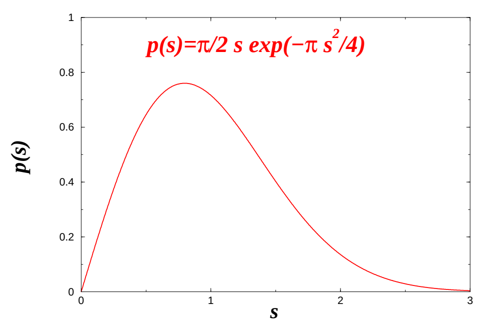
- **Berry, Tabor** (1997):
  Integrable systems $\Longrightarrow$ **Poisson statistics**





- **Bohigas, Giannoni, Schmit** (1984):
  Chaotic systems $\Longrightarrow$ **Random Matrix Statistics**





3d Anderson model

$$H = \sum_i \varepsilon_i a_i^\dagger a_i - \sum_{j=\text{adjacent to } i} a_j^\dagger a_i$$

ε_i=i.i.d.r.v. between $-W/2$ and $W/2$. $\mathbf{W_c = 16 \pm 0.5}$

- When $W < \mathbf{W_c}$ states are delocalized (**metal**) and spectral statistics = RMT
- When $W > \mathbf{W_c}$ states are localized (**insulator**) and spectral statistics = Poisson
- When $W = \mathbf{W_c}$ (**metal-insulator transition**) states have **fractal** properties and a new **intermediate** type of spectral statistics has been observed numerically **Shklovskii** (1993)

Characteristic features of intermediate statistics

- **Level repulsion at small distances** as for RMT

$$p(s) \to 0 \ \text{ when } \ s \to 0$$

- **Exponential decrease of $p(s)$ at large distances** as for Poisson

$$p(s) \sim \mathrm{e}^{-as} \ \text{ when } \ s \to \infty$$

- **Linear asymptotics of the number variance**

$$\Sigma^2(L) \equiv \langle (n(L) - \bar{n}(L))^2 \rangle \to \chi L \ \text{ when } \ L \to \infty$$

$\chi =$ spectral compressibility. $\chi = 1$ for Poisson, $\chi = 0$ for RMT

- **Multi-fractal character of eigenfunctions**

$$\langle |\Psi|^{2q} \rangle \to L^{-(q-1)D_q} \ \text{ when } \ L \to \infty$$

$D_q = 0$ for Poisson, $D_q = 1$ for RMT

Random matrix models of intermediate statistics

$$M_{ij} = \varepsilon_j \delta_{ij} + V(i-j) \ , \quad \text{typically} \ \ V(i-j) \sim \frac{g}{|i-j|^{\alpha}}$$

$\varepsilon_j =$ i.i.d.r.v. between $-W/2$ and $W/2$.

States i and j are in **resonances** provided

$$|\varepsilon_j - \varepsilon_i| \leq |V(i-j)|$$

Number of resonances connected with a cite i

$$N_{\text{resonances}}(i) \sim \sum_j |V(i-j)|$$

If

- $\alpha > 1 \Longrightarrow$ **localization**
- $\alpha < 1 \Longrightarrow$ **delocalization**
- $\alpha = 1 \Longrightarrow$ **intermediate statistics**

Critical band random matrix ensemble

$N \times N$ random matrices (e.g. Evers, Mirlin (2008)):
H_{ij} are i.i.d. Gaussian variables (real for $\beta = 1$ and complex for $\beta = 2$) with zero mean $\langle H_{ij} \rangle = 0$ and with variance

$$\langle |H_{ij}|^2 \rangle = \left(1 + \frac{(i-j)^2}{b^2} \right)^{-1}$$

Perturbation series

- $b \gg 1$: $D_q = 1 - \frac{q}{2\pi\beta b}$, $\chi = \frac{1}{2\pi\beta b}$
- $b \ll 1$:

  For $\beta = 1$

$$D_q = 2b \frac{\Gamma(2q-1)}{2^{2q-3}\Gamma(q)\Gamma(q-1)} , \quad \chi = 1 - 4b$$

  For $\beta = 2$, $b \to \frac{\pi}{2\sqrt{2}} b$

Short-range Dyson gas model

Standard RMT

$$P(\lambda_1,\lambda_2,\ldots,\lambda_N) \sim \exp\left[\beta\sum_{i<j}\ln|\lambda_j-\lambda_i| - \sum_j V(\lambda_j)\right]$$

$\beta = 1, 2, 4$ for GOE, GUE, and GSE

Short-range gas model: $\lambda_1 < \lambda_2 < \ldots < \lambda_N$

$$P(\lambda_1,\lambda_2,\ldots,\lambda_N) \sim \exp\left[\beta\sum_{j}\ln|\lambda_{j+1}-\lambda_j| - \sum_j V(\lambda_j)\right]$$

Similar for finite number of nearest levels

All correlation functions are calculated analytically

Semi-Poisson statistics

n **nearest-neighbor spacing distribution**

$$P(n,s) \sim \mathbf{s}^{\beta+\mathbf{n}(\beta+\mathbf{1})}\mathrm{e}^{-(\beta+\mathbf{1})\mathbf{s}}$$

2-point correlation formfactor

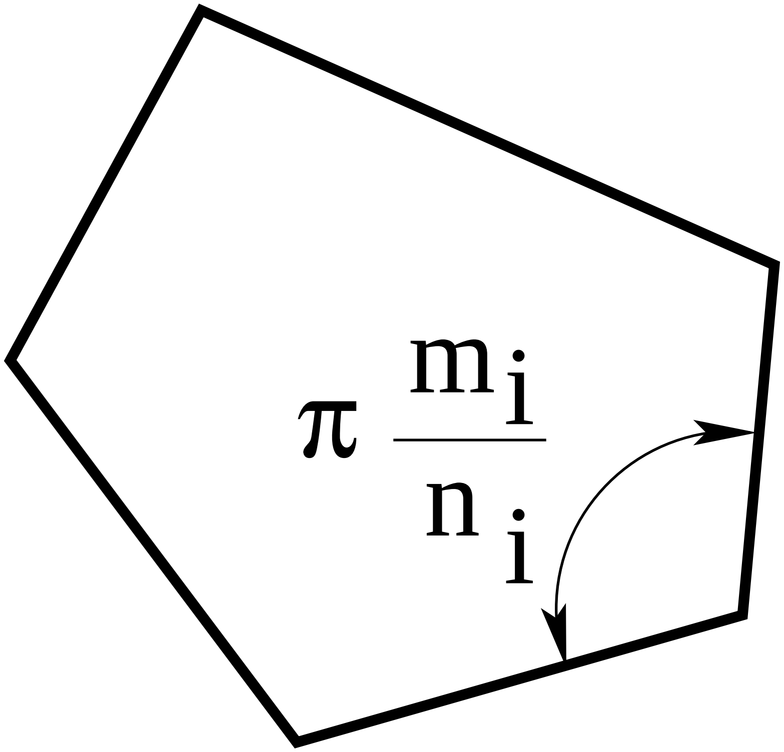
$$K(\tau) = \left[\left(1+\frac{\tau}{\beta+1}\right)^{\beta+1} - 1\right]^{-1}$$

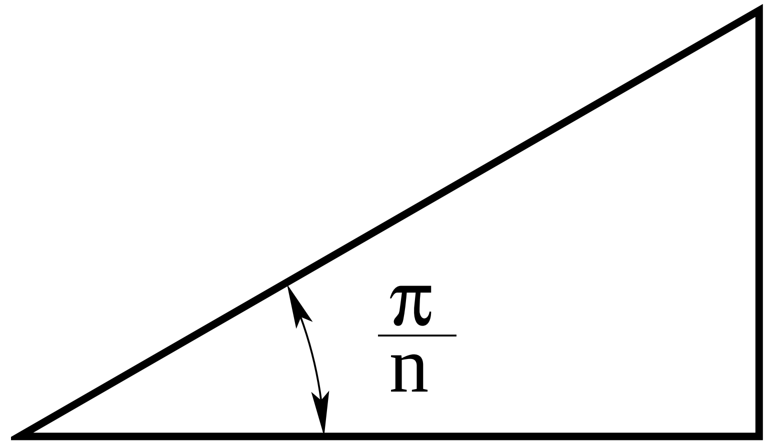
Level compressibility

$$\chi = K(0) = \frac{1}{\beta+1}$$

For $\beta = 1$, $\quad p(s) = 4s\mathrm{e}^{-2s}$, $R_2(s) = 1-\mathrm{e}^{-4s}$, $\chi = 1/2$

Polygonal billiards



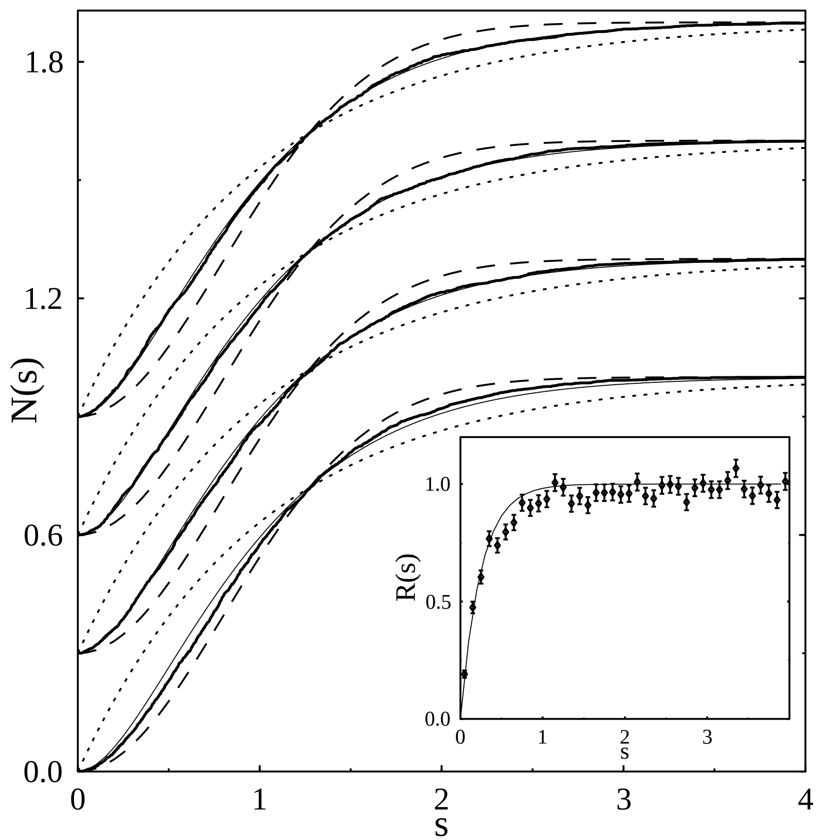


$$g = 1 + \frac{N}{2} \sum_i \frac{m_i - 1}{n_i}$$

$N =$ the least common multiple of n_i

Large variety of different behaviors

$\frac{\pi}{5}$ right triangle

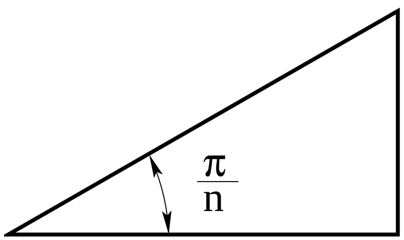


Semi-Poisson formulas:

$N_{\rm sp}(s) = 1 - (2s+1)\mathrm{e}^{-2s},\ R_2(s) = 1 - \mathrm{e}^{-4s}$

Analytical calculation of level compressibility

E.B., Giraud, Schmit (2001)

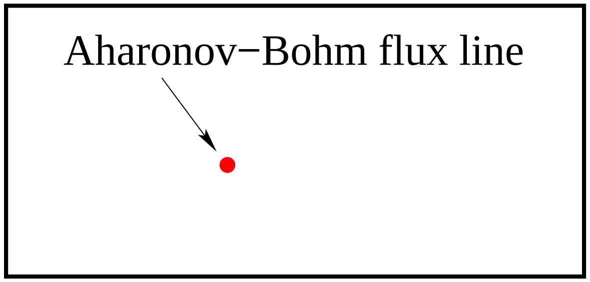


$$\chi \equiv K(0) = \frac{n + \epsilon(n)}{3(n-2)}$$

$$\epsilon(n) = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even but not divisible by 3} \\ 6 & \text{when } n \text{ is divisible by 6} \end{cases}$$

Calculations are based on the existence of the **Veech** group

Rectangular billiard with a flux line



Aharonov-Bohm flux line

$A_\phi = \frac{\alpha}{r}$ at point x_0, y_0

$\Psi_n(r,\phi) = 0$ on a rectangle a, b

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\phi} - \mathrm{i}\alpha\right)^2 + E_n\right]\Psi_n(r,\phi) = 0$$

$\tilde{\alpha}$ = fractional part of the flux

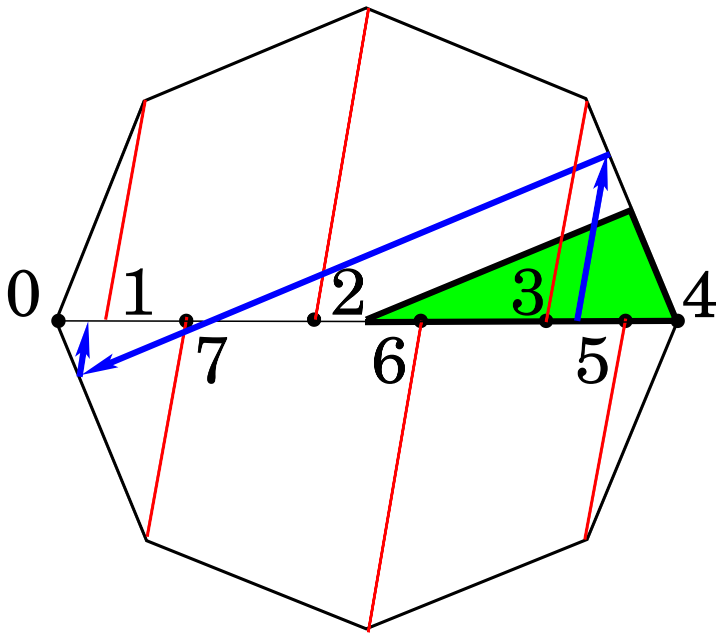
$$\chi \equiv K(0) = 1 - 4\tilde{\alpha}(1-\tilde{\alpha}) + 6\tilde{\alpha}\eta$$

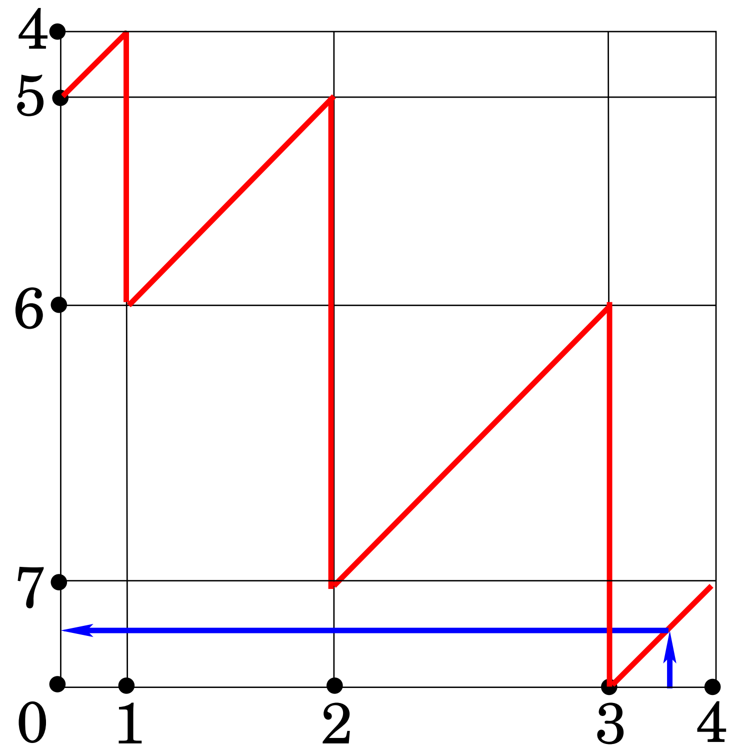
η = explicit function of $e_1 = x_0/a$ and $e_2 = y_0/b$

For irrational e_1, e_2, $\eta = 1/6$ and

$$\chi \equiv K(0) = 1 - 3\tilde{\alpha} + 4\tilde{\alpha}^2$$

Classical mechanics of pseudo-integrable billiards





Interval-exchange map: $I_1, I_2, I_3, I_4 \longrightarrow I_4, I_3, I_2, I_1$

The **simplest** interval exchange map: $(I_1, I_2) \longrightarrow (I_2, I_1)$

Consider a sequence of **parabolic** 2-dim maps

$$\Phi_0 \quad : \quad \begin{pmatrix} p \\ x \end{pmatrix} \longmapsto \begin{pmatrix} p \\ x + f(p) \end{pmatrix} \quad \text{mod } 1$$

$$\rho_\alpha \quad : \quad \begin{pmatrix} p \\ x \end{pmatrix} \longmapsto \begin{pmatrix} p + \alpha \\ x \end{pmatrix} \quad \text{mod } 1$$

$$\Phi_\alpha \quad : \quad \begin{pmatrix} p \\ x \end{pmatrix} \longmapsto \begin{pmatrix} p + \alpha \\ x + f(p + \alpha) \end{pmatrix} \quad \text{mod } 1$$

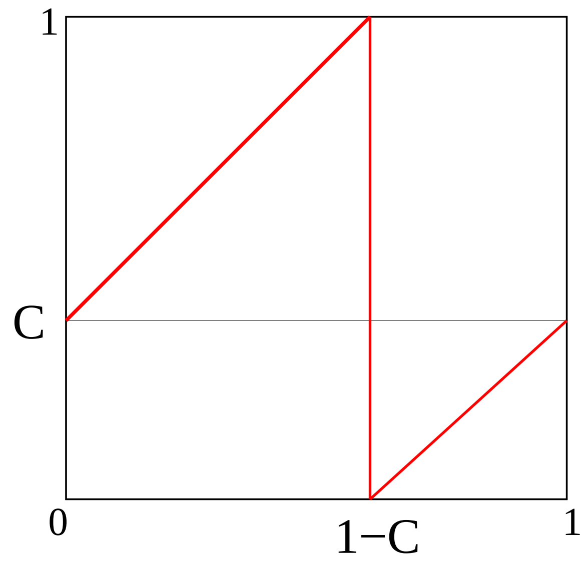
Rational $\alpha = m/q$

$\mathbf{p} = \mathbf{p} + \mathbf{k}/\mathbf{q}$ and Φ_α^q = **"pseudo-integrable"** map

$$\Phi_\alpha^q : \begin{pmatrix} p \\ x \end{pmatrix} \longmapsto \begin{pmatrix} p \\ x + C \end{pmatrix} \bmod 1 \,, \quad C = \sum_{j=1}^{q} f(p + j\alpha)$$

Simplest interval-exchange map of **two intervals** :

$\mathbf{x} \longmapsto \mathbf{x} + \mathbf{C} \bmod \mathbf{1}$



Quantization of map

$\approx$ a **unitary** matrix whose **saddle points** $=$ classical map

$$\langle Q'|U(\Phi_\alpha)|Q\rangle =$$

$$= \quad \frac{1}{N}\sum_{k=0}^{N-1} \exp[2\pi i(-N\Phi(\frac{k}{N}) + \frac{k}{N}(Q'-Q)) + 2\pi i\alpha Q]$$

$\Phi(k)' = f(p)$ - **Giraud, Marklof, O'Keefe** (2004) for $\Phi(p) = p^2$

Momentum representation

Unitary $N \times N$ **matrix :**

$$\mathbf{M_{kp}} = \mathbf{e}^{\mathbf{i}\boldsymbol{\Phi}_\mathbf{k}} \frac{\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}\alpha\mathbf{N}}}{\mathbf{N}[\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}(\mathbf{k}-\mathbf{p}+\alpha\mathbf{N})/\mathbf{N}}]} \; .$$

$$\boldsymbol{\Phi}_\mathbf{k} = -N\Phi(k/N) \; , \quad k,p = 0,1,\ldots,N-1$$

$$\mathbf{M} = \begin{pmatrix} e^{i\boldsymbol{\Phi_1}} & & & \\ & e^{i\boldsymbol{\Phi_2}} & & \\ & & \ddots & \\ & & & e^{i\boldsymbol{\Phi_N}} \end{pmatrix} \mu$$

$$\mu_{kp} = \frac{1 - e^{2\pi i \alpha N}}{N(1 - e^{2\pi i (k-p+\alpha N)/N})}$$

Two cases

- **Non - symmetric** (analog of **GUE**):

  $\boldsymbol{\Phi_k}$ are **i.i.d. random variables** with uniform distribution between 0 and 2π.

- **With 'time-reversal' symmetry** (analog of **GOE**):

  Only a **half** of coefficients is independent. The other are obtained from the symmetry : $\boldsymbol{\Phi_{N-k} = \Phi_k}$.

Main results: E.B., Schmit (2004)

For $\alpha = \mathbf{m/q}$ **and** $\mathbf{mN} \equiv \pm\mathbf{1} \bmod \mathbf{q}$ spectral statistics of the main matrix = the **semi-Poisson statistics** with parameter

$$\beta = \begin{cases} q-1 & \text{for non-symmetric matrices} \\ \frac{1}{2}(q-2) & \text{for symmetric matrices} \end{cases}$$

The nearest-neighbor distribution:

$$\mathbf{p}(\mathbf{s}) = \mathbf{A}_{\beta}\mathbf{s}^{\beta}\mathrm{e}^{-(\beta+\mathbf{1})\mathbf{s}}$$

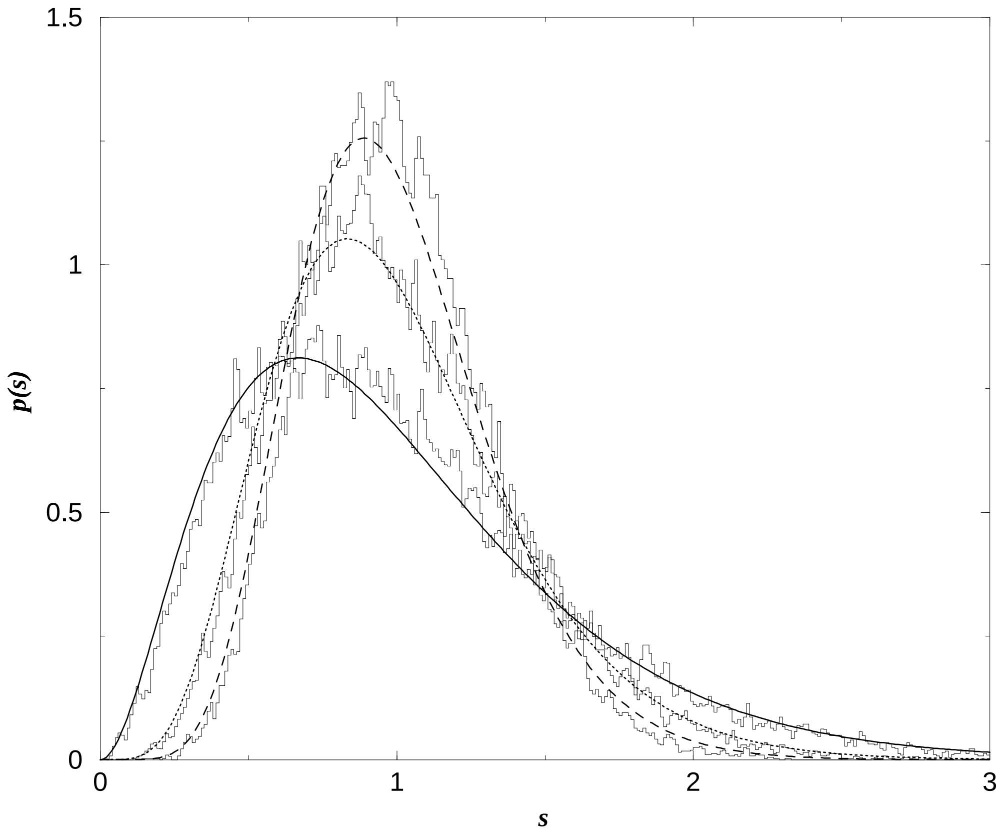
Dyson's dogma:

In random matrices one gets level repulsion ($\mathbf{p}(\mathbf{s}) \sim \mathbf{s}^{\beta}$) with **only** three values of β

$$\beta = \mathbf{1}\ ,\ \mathbf{2}\ ,\ \mathbf{4}.$$

In **pseudo-integrable** maps it is **not correct**.

Non-symmetric matrices



Solid line: $\alpha = 1/3$

$p(s) \sim s^2 \, \mathrm{e}^{-3s}$

Dotted line: $\alpha = 1/6$

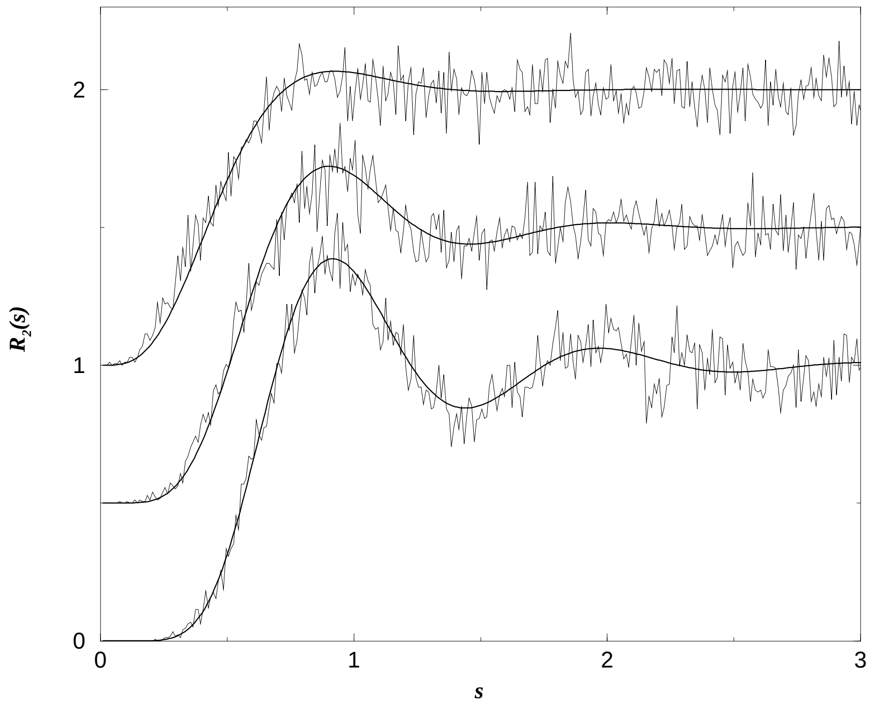
$p(s) \sim s^5 \, \mathrm{e}^{-6s}$

Dashed line: $\alpha = 1/9$

$p(s) \sim s^8 \, \mathrm{e}^{-9s}$

The **nearest-neighbor distribution** for $\alpha = 1/3, 1/6, 1/9$.

Non-symmetric matrices

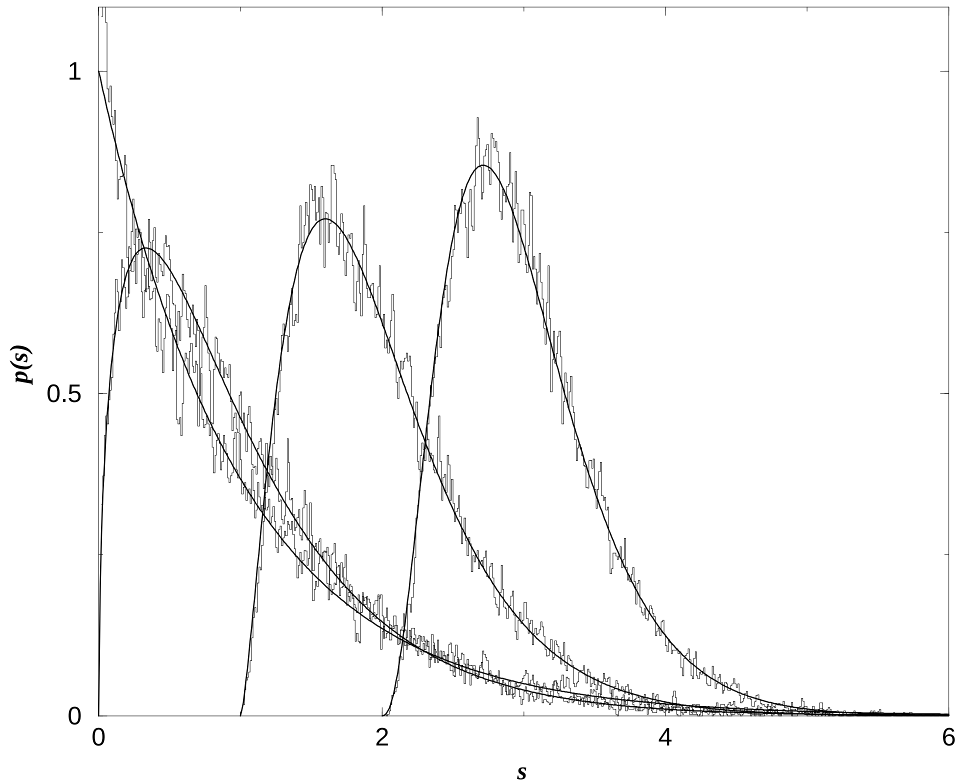


For integer β:

$$R_2^{(\beta)}(s) = \mathrm{e}^{-(\beta+1)s} \sum_{k=0}^{\beta} \exp[(\beta + 1)s\mathrm{e}^{2\pi \mathrm{i}k/(\beta+1)} + \mathrm{e}^{2\pi \mathrm{i}k/(\beta+1)}]$$

The **two-point correlation function** for $\alpha = 1/4, 1/7, 1/10$.

Symmetric matrices



The **nearest-neighbor distribution** for symmetric matrices with $\alpha = 1/2, 1/3, 1/5, 1/7$.

- $\alpha = 1/2$:
  $p(s) = \mathrm{e}^{-s}$
- $\alpha = 1/3$:
  $p(s) \sim s^{1/2}\ \mathrm{e}^{-3/2s}$
- $\alpha = 1/5$:
  $p(s) \sim s^{3/2}\ \mathrm{e}^{-5/2s}$
- $\alpha = 1/7$:
  $p(s) \sim s^{5/2}\mathrm{e}^{-7/2s}$

Spectral statistics when $N \neq \pm 1$ mod q

For $\alpha = m/q$ and $N \equiv -k \bmod q$ with $k = 1, \ldots, q-1$ correlation functions are calculated from a **transfer matrix** of dimension C_{q-2}^{k-1} (E.B., Dubertand, Schmit (2008))

Example: For **non-symmetric matrices** with $\alpha = 1/5$ and $N \equiv \pm 2 \bmod 5$ the transfer matrix is

$$T(x) = \begin{pmatrix} 3\frac{x^4}{4!} & 5\frac{x^5}{5!} & 5\frac{x^6}{6!} \\ 3\frac{x^3}{3!} & 5\frac{x^4}{4!} & 5\frac{x^5}{5!} \\ 2\frac{x^2}{2!} & 3\frac{x^3}{3!} & 3\frac{x^4}{4!} \end{pmatrix} \exp(-5x).$$

$$\boxed{p(s) = (a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6) e^{-5s}}$$

$a_2 = 625/2 - 275\sqrt{5}/2 \approx 5.041$, $a_3 = 3125/2 - 1375\sqrt{5}/2 \approx 25.203$,

$a_4 = 71875/48 + 33125\sqrt{5}/48 \approx 45.724$,

$a_5 = -15625/3 + 9375\sqrt{5}/4 \approx 32.451$,

$a_6 = 1015625/288 - 453125\sqrt{5}/288 \approx 8.357$.

Transfer matrix for $\alpha = 1/7$ and $N \equiv \pm 3 \bmod 7$

For **non-symmetric** matrices: $p_r \sim x^r \mathrm{e}^{-x}$

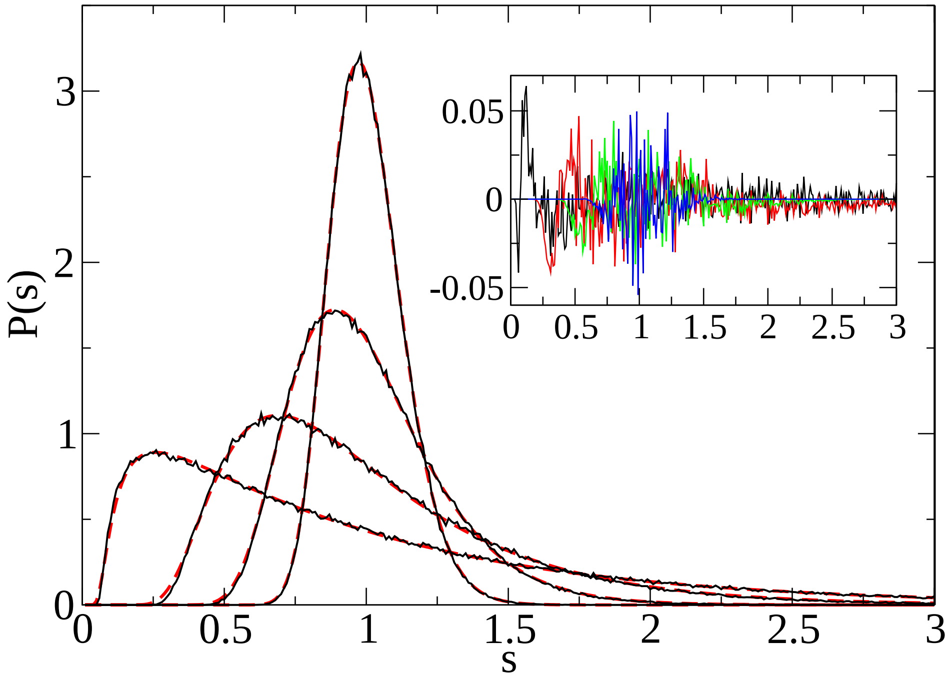
For **symmetric** matrices: $p_r \sim x^{(r-1)/2} \mathrm{e}^{-x/2}$

$$\begin{pmatrix}
10p_6 & 35p_7 & 70p_8 & 84p_9 & 56p_8 & 168p_9 & 252p_{10} & 210p_{10} & 462p_{11} & 462p_{12} \\
10p_5 & 35p_6 & 70p_7 & 84p_8 & 56p_7 & 168p_8 & 252p_9 & 210p_9 & 462p_{10} & 462p_{11} \\
6p_4 & 20p_5 & 40p_6 & 49p_7 & 30p_6 & 91p_7 & 140p_8 & 112p_8 & 252p_9 & 252p_{10} \\
3p_3 & 8p_4 & 15p_5 & 19p_6 & 10p_5 & 30p_6 & 49p_7 & 35p_7 & 84p_8 & 84p_9 \\
4p_4 & 15p_5 & 30p_6 & 35p_7 & 26p_6 & 77p_7 & 112p_8 & 98p_8 & 210p_9 & 210p_{10} \\
3p_3 & 12p_4 & 25p_5 & 30p_6 & 20p_5 & 61p_6 & 91p_7 & 77p_7 & 168p_8 & 168p_9 \\
2p_2 & 6p_3 & 12p_4 & 15p_5 & 8p_4 & 25p_5 & 40p_6 & 30p_6 & 70p_7 & 70p_8 \\
0 & 3p_3 & 8p_4 & 10p_5 & 6p_4 & 20p_5 & 30p_6 & 26p_6 & 56p_7 & 56p_8 \\
0 & 2p_2 & 6p_3 & 8p_4 & 3p_3 & 12p_4 & 20p_5 & 15p_5 & 35p_6 & 35p_7 \\
0 & 0 & 2p_2 & 3p_3 & 0 & 3p_3 & 6p_4 & 4p_4 & 10p_5 & 10p_6
\end{pmatrix}$$

Another example

$$\mathbf{L_{kr} = p_r \delta_{kr} + i\left(\frac{a}{2}\right)\frac{1-\delta_{kr}}{k-r}}$$

p_r = i.i.d.r.v. uniform between -1 and 1, $k, r = -N/2, \dots N/2$



$$P(s) = A\mathrm{e}^{-B^2/s^2 - Cs}$$

$a = .1,\ .5,\ 1,\ 2,\ N = 301$

Difficulties with intermediate-type ensembles

- Matrices and related physical problems are **not** invariant over the basis change
- Analytical results are rare and one has rely on numerics
- Large variety of different behaviors and absence of universality

Classical integrable systems

Calogero-Moser models

Inr
$$H(p,q) = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + a^2 \sum_{1 \le j < k \le N} \frac{1}{(q_j - q_k)^2}$$

IInr
$$H(p,q) = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \frac{1}{4} a^2 \mu^2 \sum_{1 \le j < k \le N} \frac{1}{\sinh^2(\frac{\mu}{2}(q_j - q_k))}$$

IIInr
$$H(p,q) = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \frac{1}{4} a^2 \mu^2 \sum_{1 \le j < k \le N} \frac{1}{\sin^2(\frac{\mu}{2}(q_j - q_k)/2)}$$

Ruijsenaars-Schneider model

IIIb
$$H(p,q) = \sum_{j=1}^{N} \cos(p_j) \prod_{k \neq j} \left(1 - \frac{\sin^2 \pi a}{\sin^2 \frac{\mu}{2}(q_j - q_k)} \right)^{1/2}$$

Integrability

A N-dim system = integrable, if $\exists\ N$ integrals of motion $I_j(\vec{p},\vec{q}\,)$

$$\{I_j(\vec{p},\vec{q}\,), I_k(\vec{p},\vec{q}\,)\} = 0$$

Angle-action variables: $I_j = I_j(\vec{p},\vec{q}\,)$, $\phi_j = \phi_j(\vec{p},\vec{q}\,)$

$$\dot{I}_j(\vec{p},\vec{q}\,) = 0\ ,\quad \dot{\phi}_j = \omega_j(\vec{I}(\vec{p},\vec{q}\,))$$

Canonicity:

$$\boxed{\mathrm{d}\vec{\mathbf{p}}\,\mathrm{d}\vec{\mathbf{q}} = \mathrm{d}\vec{\mathbf{I}}\,\mathrm{d}\vec{\phi}}$$

Lax matrices

Pair of matrices $\mathbf{L_{kr}}(\vec{\mathbf{p}},\vec{\mathbf{q}}\,)$ and $\mathbf{M_{kr}}(\vec{\mathbf{p}},\vec{\mathbf{q}}\,)$ such that the equations of motion are consequence of the Lax equation

$$\dot{\mathbf{L}}(\vec{\mathbf{p}},\vec{\mathbf{q}}\,) = [\mathbf{L}(\vec{\mathbf{p}},\vec{\mathbf{q}}\,)\ ,\ \mathbf{M}(\vec{\mathbf{p}},\vec{\mathbf{q}}\,)]$$

Lax matrices for Calogero-Moser models

- **Rational**

$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mathbf{1} - \delta_{\mathbf{kr}}}{\mathbf{q_k} - \mathbf{q_r}}$$

- **Hyperbolic**

$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sinh(\mu(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2})}$$

- **Trigonometric**

$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sin(\mu(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2})}$$

Lax matrix for Ruijsenaars-Schneider model

$$\mathbf{L_{kp}} = \mathrm{e}^{\mathrm{i}\mathbf{p_k} + \mathrm{i}(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2}} \mathbf{C_{kp}}(\mathbf{a}; \vec{\mathbf{q}})$$

$C(\mathbf{a}; \vec{\mathbf{q}})$ is an **orthogonal** matrix $(C \cdot C^t = 1)$

$$C_{kp}(\mathbf{a}; \vec{\mathbf{q}}) = W_k^{1/2}(\mathbf{a}; \vec{\mathbf{q}}) \frac{\sin \pi \mathbf{a}}{\sin\left(\frac{\mathbf{q_k} - \mathbf{q_p}}{2} + \pi \mathbf{a}\right)} W_p^{1/2}(-\mathbf{a}; \vec{\mathbf{q}}) \ .$$

where

$$W_j(\mathbf{a}, \vec{\mathbf{q}}) = \prod_{s \neq j} \frac{\sin\left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2} + \pi \mathbf{a}\right)}{\sin\left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2}\right)}$$

General construction

Lax matrix $L(\vec{\mathbf{p}}, \vec{\mathbf{q}})$ = a random matrix depending on random variables $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ distributed according to a "**natural**" measure

$$\mathrm{d}L = P(\vec{\mathbf{p}}, \vec{\mathbf{q}})\, \mathrm{d}\vec{\mathbf{p}}\, \mathrm{d}\vec{\mathbf{q}}$$

Integrability: canonical action-angle variables $I_\alpha(\vec{\mathbf{p}}, \vec{\mathbf{q}})$ and $\phi_\alpha(\vec{\mathbf{p}}, \vec{\mathbf{q}})$: $\boxed{\prod_j \mathrm{d}p_j\, \mathrm{d}q_j = \prod_\alpha \mathrm{d}I_\alpha\, \mathrm{d}\phi_\alpha\ .}$

Usually $I_\alpha(\vec{\mathbf{p}}, \vec{\mathbf{q}})$ = eigenvalues λ_α of the Lax matrix or a simple function of them. The canonical change of variables

$$\mathrm{d}L = \mathcal{P}(\vec{\lambda}, \vec{\phi})\, \mathrm{d}\,\vec{\lambda}\, \mathrm{d}\,\vec{\phi}$$

The **exact** joint distribution of eigenvalues

$$P(\vec{\lambda}) = \int \mathcal{P}(\vec{\lambda}, \vec{\phi})\, \mathrm{d}\,\vec{\phi}$$

This scheme can be adapted to many different models

Angle-action variables for the Calogero model

$$\mathbf{L_{kr} = p_r \delta_{kr} + ig \frac{1 - \delta_{kr}}{q_k - q_r}}$$

$$\sum_r L_{kr} u_r(n) = \lambda_n u_k(n), \quad \sum_m u_k^*(m) u_r(m) = \delta_{kr}$$

Direct calculations: $\qquad L_{kr} q_r - q_k L_{kr} = -\mathrm{i}g(1 - \delta_{kr})$.

$$Q_{mn}(\lambda_m - \lambda_n) = -\mathrm{i}g(e_m^* e_n - \delta_{mn})$$

$$Q_{mn} = \sum_k u_k^*(m) q_k u_k(n), \; e_m = \sum_k u_k(m), \; \sum_n Q_{mn} u_k^*(n) = q_k u_k^*(m)$$

One can choose $e_m = 1$. Then

$$\mathbf{Q_{mn} = w_m \delta_{mn} - ig \frac{1 - \delta_{mn}}{\lambda_m - \lambda_n}}$$

Ruijsenaars proved that $w_m = \phi_m$ are **angle variables** and $\lambda_m = I_m$ are **action variables**

Natural measure of random ensemble

Consider L_{kr} as a random matrix depending on p and q with the **natural** measure

$$\mathrm{d}L \sim \exp\left[-\alpha \mathrm{Tr} L^\dagger L - \beta \sum_k q_k^2\right] \mathrm{d}p\mathrm{d}q$$

$$\equiv \exp\left[-\alpha \left(\sum_k p_k^2 + g^2 \sum_{i\neq j} \frac{1}{(q_i - q_j)^2}\right) - \beta \sum_k q_k^2\right] \mathrm{d}p\mathrm{d}q$$

In variables λ and w this distribution can be rewritten as

$$\mathrm{d}L \sim \exp\left[-\alpha \sum_m \lambda_m^2 - \beta \mathrm{Tr} Q^\dagger Q\right] \mathrm{d}\lambda\mathrm{d}w$$

$$\equiv \exp\left[-\alpha \sum_m \lambda_m^2 - \beta \left(\sum_m w_m^2 + g^2 \sum_{m\neq n} \frac{1}{(\lambda_m - \lambda_n)^2}\right)\right] \mathrm{d}\lambda\mathrm{d}w \ .$$

Exact joint distribution of eigenvalues for Calogero-Moser ensemble

$$P(\lambda_1,\ldots,\lambda_N)\sim \exp\left[-\alpha\sum_m \lambda_m^2-\beta g^2\sum_{m\neq n}\frac{1}{(\lambda_m-\lambda_n)^2}\right] .$$

Wigner-type surmise for the nearest-neighbor distribution

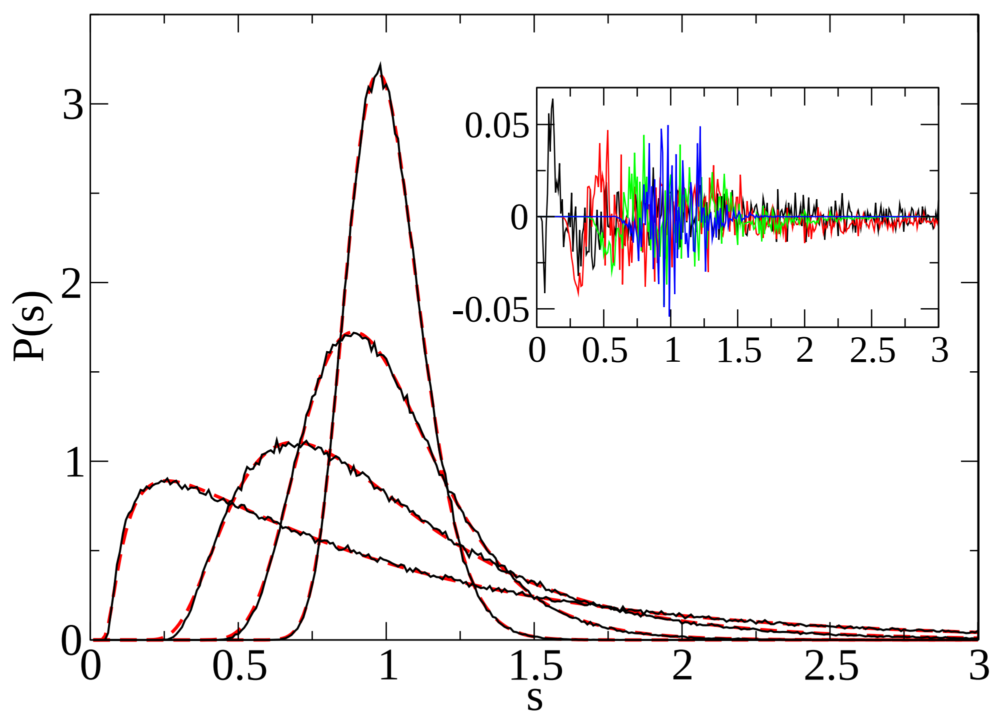
$$\mathbf{p(s)=A}\mathrm{e}^{-\mathbf{B^2/s^2-Cs}}$$

In thermodynamic limit when $N\to\infty$ coordinates q_k are in a box of length $L\to\infty$ with $N/L=$ constant the exact Lax matrix can be simplified by fixing $q_k\sim k$

$$\tilde{\mathbf{L}}_{\mathbf{jk}}=\mathbf{p_k}\delta_{\mathbf{jk}}+\mathrm{i}\mathbf{a}\frac{\mathbf{1}-\delta_{\mathbf{jk}}}{\mathbf{2(j-k)}}$$

Here p_k are i.i.d. random variables with uniform distribution between -1 and 1 and j,k are integers from -$N/2$ till $N/2$

Numerics for Calogero-Moser ensemble

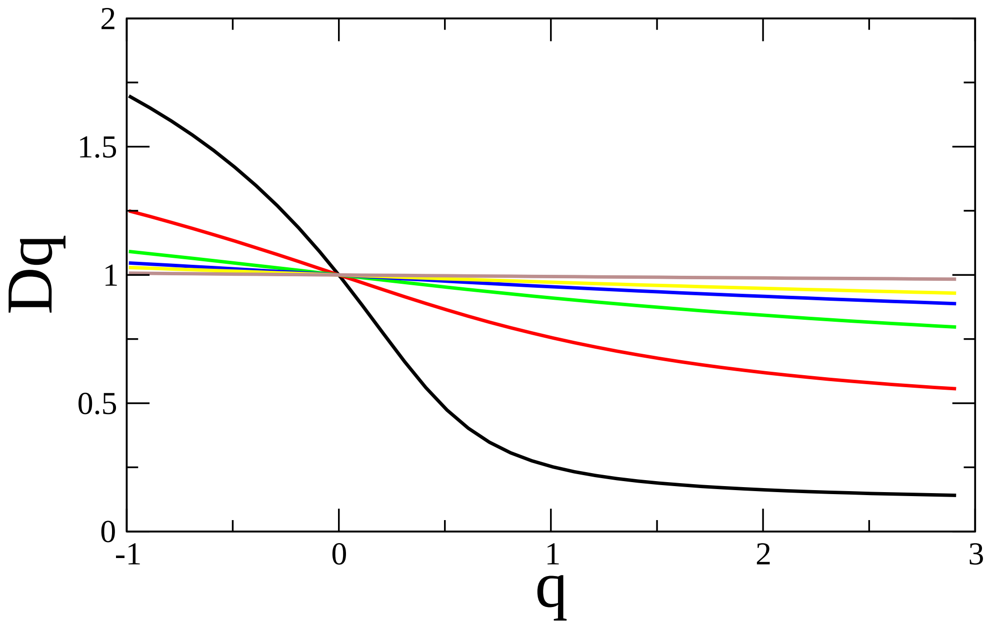


Wigner-type surmise $\mathbf{p(s) = A}\mathrm{e}^{\mathbf{-B^2/s^2 - Cs}}$

Fit $B = .096,\ .618,\ 1.46,\ 3.11$ for $a = .1,\ .5,\ 1,\ 2$

Fractal properties of eigenfunctions for Calogero-Moser ensemble

$$\langle |\Psi|^{2q} \rangle \to N^{-(q-1)D_q} \quad \text{when} \quad N \to \infty$$



$$a = 0.1, 0.5, 1.0, 1.5, 2.0, 5.0$$

Angle-action variables for Ruijsenaars model

$$\mathbf{L_{kp}} = \mathrm{e}^{\mathrm{i}\mathbf{p_k} + \mathrm{i}(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2}} \mathbf{C_{kp}}(\mathbf{a}, \vec{\mathbf{q}})$$

$$\mathbf{C_{kp}}(\mathbf{a}, \vec{\mathbf{q}}) = W_k^{1/2}(\mathbf{a}; \vec{\mathbf{q}}) \frac{\sin \pi \mathbf{a}}{\sin\left(\frac{\mathbf{q_k} - \mathbf{q_p}}{2} + \pi \mathbf{a}\right)} W_p^{1/2}(-\mathbf{a}, \vec{\mathbf{q}})$$

$$W_j(\mathbf{a}, \vec{\mathbf{q}}) = \prod_{s \neq j} \frac{\sin\left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2} + \pi \mathbf{a}\right)}{\sin\left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2}\right)}$$

$$\textstyle\sum_{p=1}^{N} L_{kp} u_p(\gamma) = \lambda_\gamma u_k(\gamma), \ \lambda_\alpha = \mathrm{e}^{\mathrm{i}\theta_\alpha},$$

$$\boxed{\mathbf{Q}_{\gamma\xi} = \textstyle\sum_{n=1}^{N} u_n(\gamma) \mathrm{e}^{\mathrm{i}q_n} u_n^*(\xi)}$$

$$\mathbf{Q}_{\gamma\xi} = \mathrm{e}^{\mathrm{i}\phi_\gamma + \mathrm{i}(\theta_\gamma - \theta_\xi)/2} \mathbf{C}_{\gamma\xi}(-\mathbf{a}, \vec{\theta})$$

When L is unitary, θ_α and ϕ_α = action-angle variables

Natural measure for Ruijsenaars model

R-S Hamiltonian is self-adjoint and the Lax matrix is unitary not on the whole $\vec{\mathbf{q}}$-space but only on a subset of it when for all j

$$V_j(\mathbf{a}, \vec{\mathbf{q}}) \equiv \prod_{k \neq j} \left(1 - \frac{\sin^2 \pi \mathbf{a}}{\sin^2[(\mathbf{q_j} - \mathbf{q_k})/2]} \right) = W_j(\mathbf{a}, \vec{\mathbf{q}}) W_j(-\mathbf{a}, \vec{\mathbf{q}}) > 0$$

$R(\mathbf{a}, \vec{\mathbf{q}}) =$ the characteristic function of this subset

$$R(\mathbf{a}, \vec{\mathbf{q}}) = \begin{cases} 1 & \text{when } V_j(\mathbf{a}, \vec{\mathbf{q}}) > 0 \ , \ j = 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

"**Natural**" measure for the RS ensemble = the **uniform** measure

$$\mathrm{d}L \sim R(\mathbf{a}, \vec{\mathbf{q}}) \, \mathrm{d}\, \vec{\mathbf{p}} \, \mathrm{d}\, \vec{\mathbf{q}}$$

By transforming this expression to action-angle variables one gets

$$P(\vec{\theta}) \sim R(\mathbf{a}, \vec{\theta})$$

Main lemma

$$\mathbf{L_{kp}} = \mathrm{e}^{\mathrm{i}\mathbf{p_k}+\mathrm{i}(\mathbf{q_k}-\mathbf{q_r})/\mathbf{2}}\mathbf{C_{kp}}(\mathbf{a},\vec{\mathbf{q}}) \ ,$$

$$\mathbf{C_{kp}}(\mathbf{a},\vec{\mathbf{q}}) = W_k^{1/2}(\mathbf{a},\vec{\mathbf{q}})\frac{\sin\pi\mathbf{a}}{\sin\left(\frac{\mathbf{q_k}-\mathbf{q_p}}{2}+\pi\mathbf{a}\right)}W_p^{1/2}(-\mathbf{a},\vec{\mathbf{q}})$$

If this matrix is unitary then its eigenvalues are such that after the rotation by $\pm\mathbf{2}\pi\mathbf{a}$ *in-between of any pairs of nearest eigenvalues there exist one and only one rotated eigenvalue*

Identity

$$\mathrm{e}^{\pm\mathrm{i}(q_k-q_p+2\pi a)} = 1 \pm \mathrm{e}^{\pm\mathrm{i}((q_k-q_p)/2+\pi a)}\sin((q_k-q_p)/2+\pi a)$$

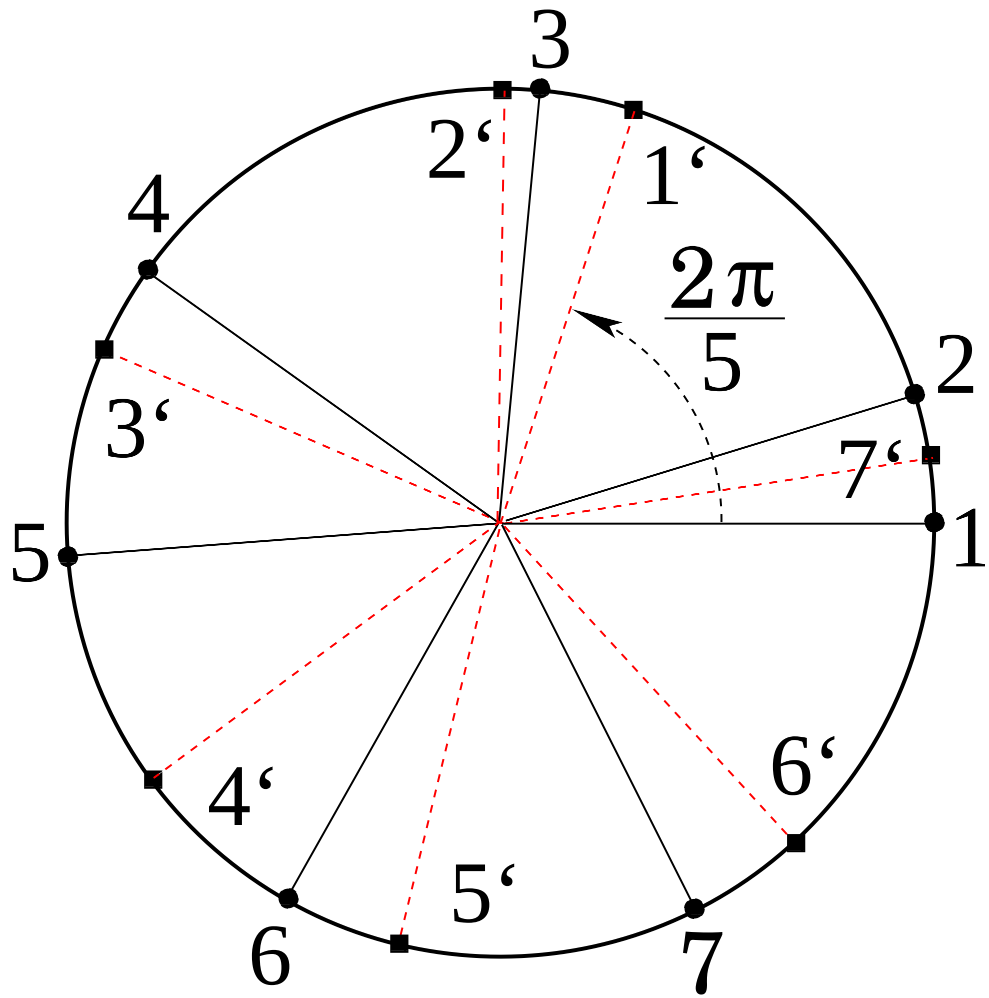
Two **rank-one** deformations

$$N_{kp}^{(\pm)} = L_{kp}\mathrm{e}^{\pm\mathrm{i}(q_k-q_j+2\pi a)}$$

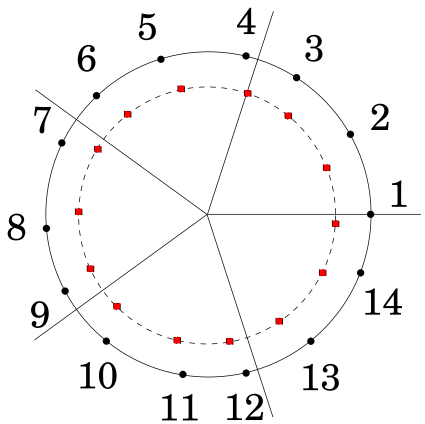
with **known** eigenfunctions and eigenvalues

$$N_{kp}^{(\pm)}\Psi_p^{(\pm)} = \Lambda_\alpha^{(\pm)}\Psi_k^{(\pm)} \ , \ \Psi_k^{(\pm)} = \mathrm{e}^{\pm\mathrm{i}q_k}u_k(\alpha) \ , \ \Lambda_\alpha^{(\pm)} = \mathrm{e}^{\pm 2\pi\mathrm{i}a}\lambda_\alpha$$

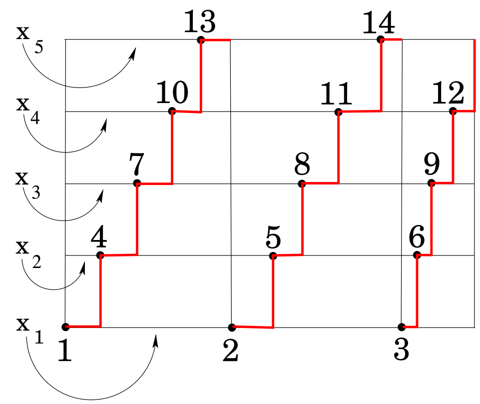
$N = 7$, $a = 1/5$



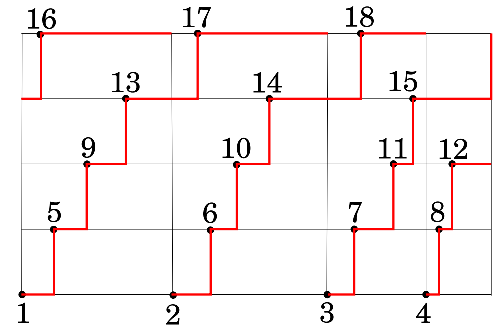
Geometrical unfolding



$a = 1/5,\ N = 14$



$N \equiv -1 \bmod q$



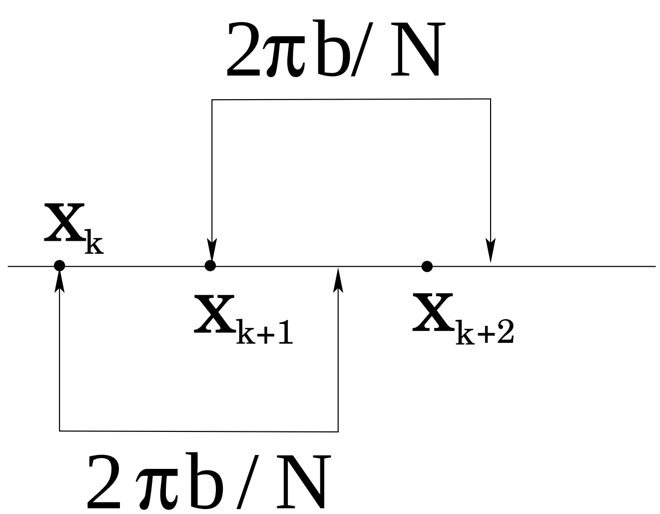
$N \equiv -2 \bmod q$

Lemma: *For $\alpha = m/q$ and $mN \equiv -k \mod q$ with $k = 1, \ldots, q-1$ eigenphases of matrix L_{kp} can be described as follows. Fix q horizontal lines, put arbitrary points at the lowest line. Draw staircase non-intersecting lines going only up and to the right with the condition that they start at the lower line and end at last line but with the shift by k units. Points at horizontal lines are situated at the corners of the constructed lines.*

Variation: $a \Longrightarrow b/N$

Lemma: *When $\alpha = b/N$ and $N > N_*$ at the angular distance of $2\pi b/N$ from each eigenvalue there exist exactly* $[\mathbf{b}]$ *eigenvalues*

- When $0 < b < 1$ the **minimal** distance between 2 eigenvalues is $2\pi b/N$.
- When $b > 1$ the **maximal** distance between 2 eigenvalues is $2\pi b/N$. Example: $\mathbf{1 < b < 2}$



Transfer operator

Let $\theta_1 < \theta_2 < \ldots < \theta_N$, $\xi_k = \theta_{k+1} - \theta_k$ and

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < b \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = 1 - f(x),$$

Joint probability of RS eigenphases inside an interval Δ

$$P(\xi) \sim \prod_{j=1}^{N} f(s_j) g(s_j + \xi_{j+n}) \delta(\Delta - \sum_{k=1}^{N} \xi_k) \; ,$$

where $s_j = \xi_j + \ldots + \xi_{j+n-1}$ and $n = [b]$.

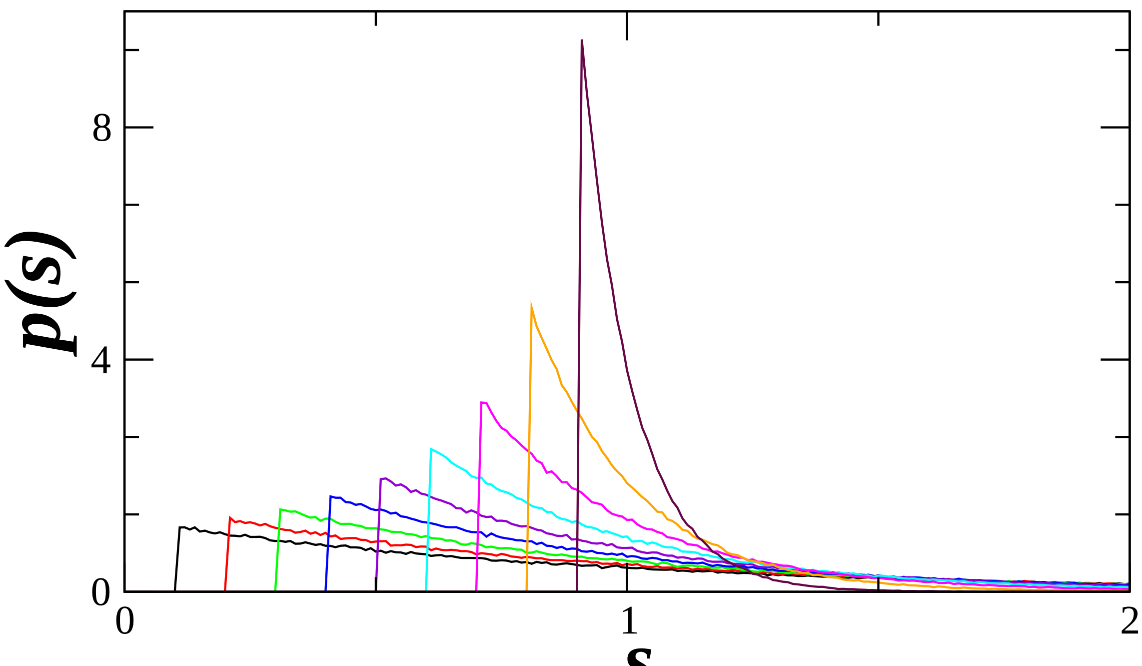
Exactly as a 1-d gas where each particle interacts with $n = [b]$ nearest-neighbors. Solution by **transfer operator**

In **thermodynamic limit**: $q_k \longrightarrow 2\pi k/N$

$$L_{kp} \longrightarrow \mathrm{e}^{\mathrm{i} p_k} \frac{1 - \mathrm{e}^{2\pi \mathrm{i} \mathbf{b}}}{N[1 - \mathrm{e}^{2\pi \mathrm{i}(k - p + \mathbf{b})/N}]}$$

$\mathbf{0 < b < 1}$

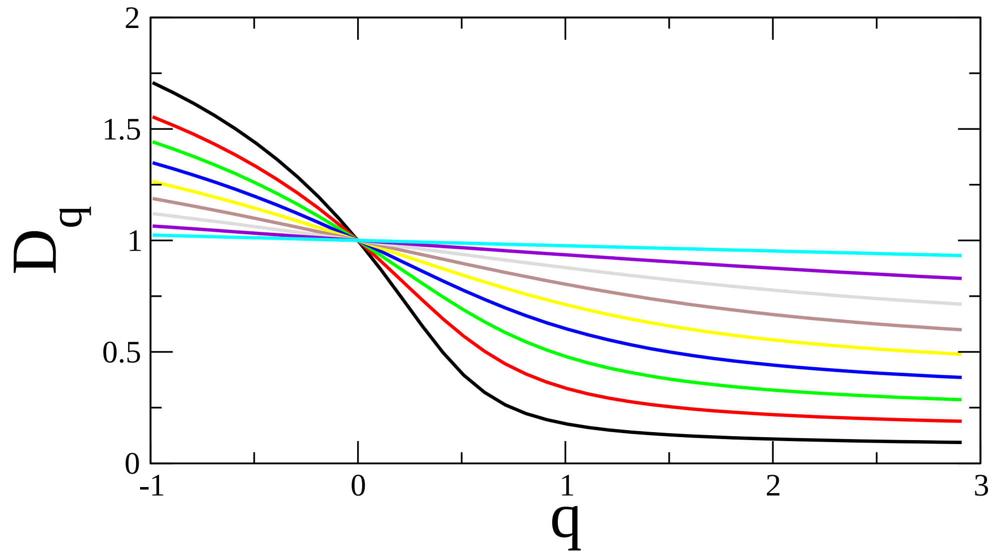
Poisson distribution shifted by b



$$b = 0.1, 0.2, \ldots, 0.9$$

Fractal properties for $0 < b < 1$

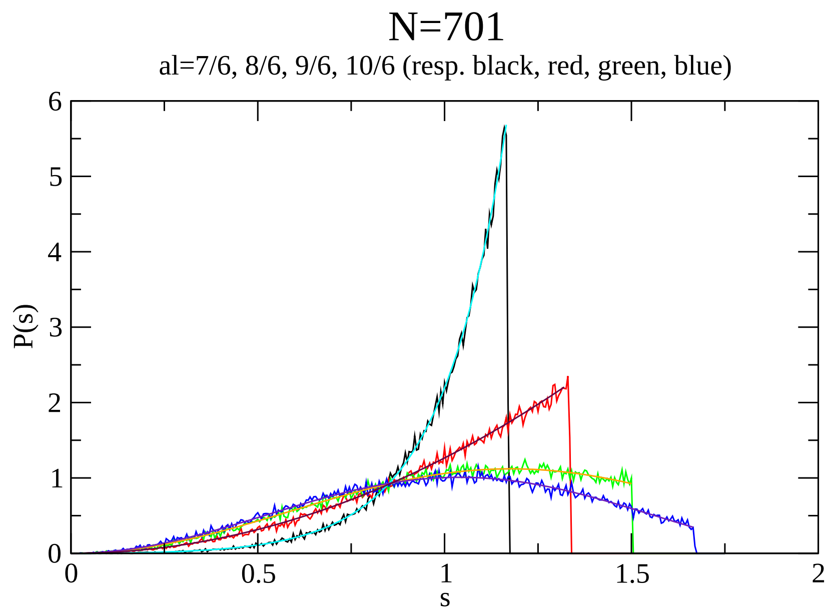
$$\langle|\Psi|^{2q}\rangle \to N^{-(q-1)D_q} \ \text{ when } \ N \to \infty$$



$$b = 0.1, 0.2, \ldots, 0.9$$

$\mathbf{1 < b < 2}$

$$p(s) = \begin{cases} A\sinh^2(\rho s) & \text{when } 1 < b < 4/3 \\ \frac{81}{64} s^2 & \text{when } b = 4/3 \\ A\sin^2(\rho s) & \text{when } 4/3 < b < 2 \end{cases}$$

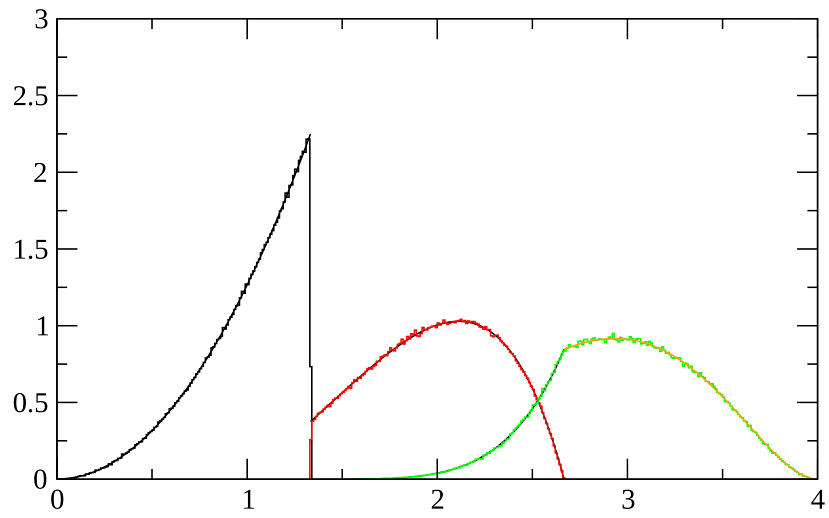


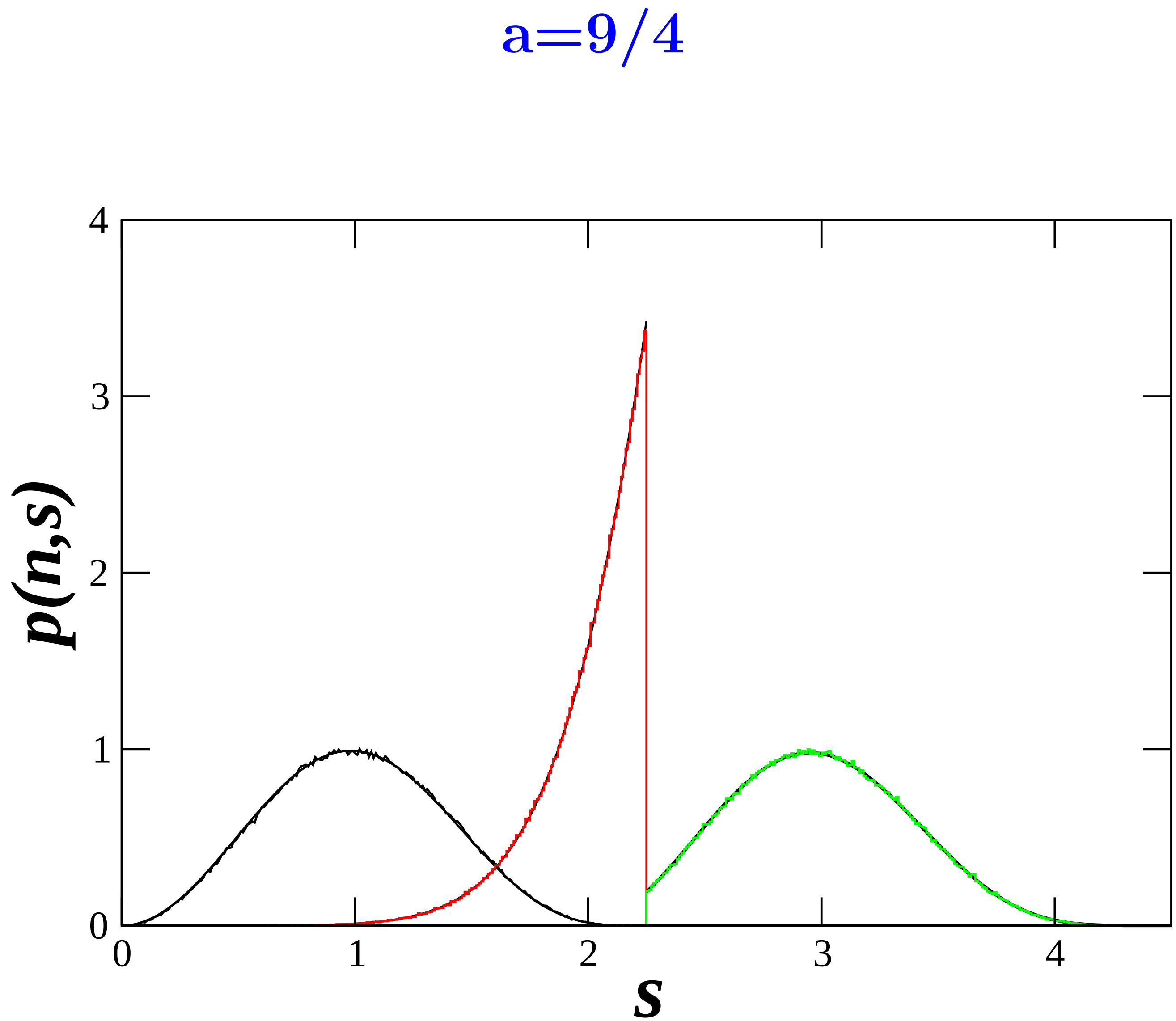
$\mathbf{b = 4/3}$

$$p(s) = \frac{81}{64}s^2 \;, \quad 0 < s < 4/3$$

$$p(2,s) = (-\frac{3}{2} + \frac{27}{16}s - \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} \;, \quad 4/3 < s < 8/3$$

$$p(3,s) = \begin{cases} (\frac{3}{4} - \frac{81}{32}s + \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} + \frac{81}{64}s^2, & 4/3 < s < 8/3 \\ (-\frac{9}{4} + \frac{27}{32}s - \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} + 9\mathrm{e}^{3s/2-4}, & 8/3 < s < 4 \end{cases}$$





a=9/4
p(n,s)
s
0
1
2
3
4
0
1
2
3
4

Summary

- Physical problems giving rise to intermediate statistics
  - Anderson model at MIT
  - **Pseudo-integrable billiards**
  - Integrable systems with flux line
  - ...
- Large varieties of intermediate statistics
  Absence of universality
- **Lax matrices** of integrable classical systems give new soluble ensembles of random matrices with intermediate statistics
- Fractal properties of eigenfunctions for investigated models
- New perspectives for intermediate statistics