FONCTION ζ DE RIEMANN ET THEORIE DE MATRICES ALEATOIRES : QUELQUES ASPECTS

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Paris, IHP, 17-18 Juin 2009

- Introduction

- Zeros de ζ et RMT (Random Matrix Theory)

- Fondement dynamique de RMT. Théorie d'orbites périodiques

- $\zeta(s)$ et théorie d'orbites périodiques

- autocovariances d'espacements

- 'Riemannium'

- vers l'asymptotique des espacements de zeros

- Conclusions

O. Bohigas, P. Leboeuf, M.-J. Sánchez

'Spectral spacing correlations for chaotic and disordered systems'

Found. of Physics 31 (2001) 489-517

P. Leboeuf, A. G. Monastra, O. Bohigas

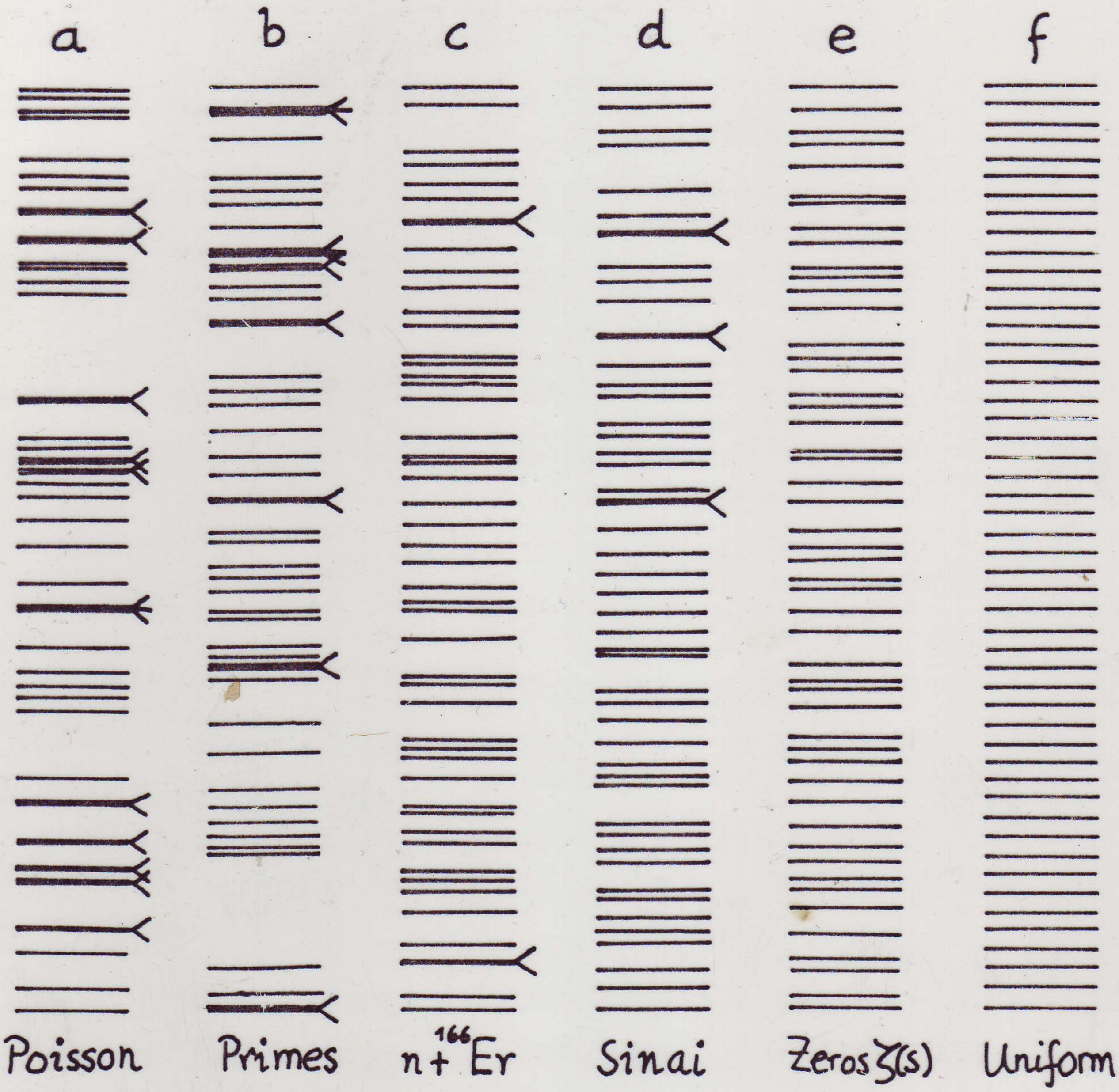
'The Riemannium'

Reg. Chaot. Dyn. 6 (2001) 205

E. Bogomolny, O. Bohigas, P. Leboeuf, A. G. Monastra

'On the spacing distribution of the Riemann zeros: corrections to the asymptotic result'

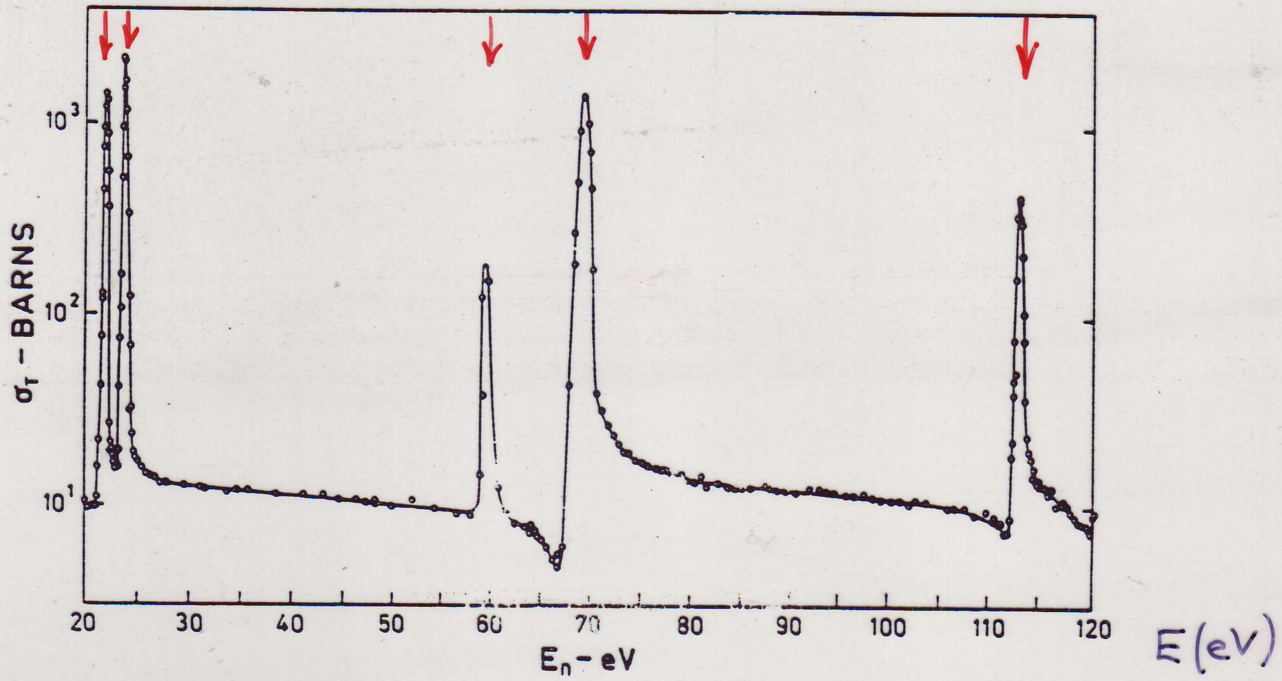
J. Phys. A 39 (2006) 10743-10754



Bohigas
Giannoni '84, Lecture Notes in Physics 209

Neutron cross section $\sigma(E)$

$n + {}^{232}Th$



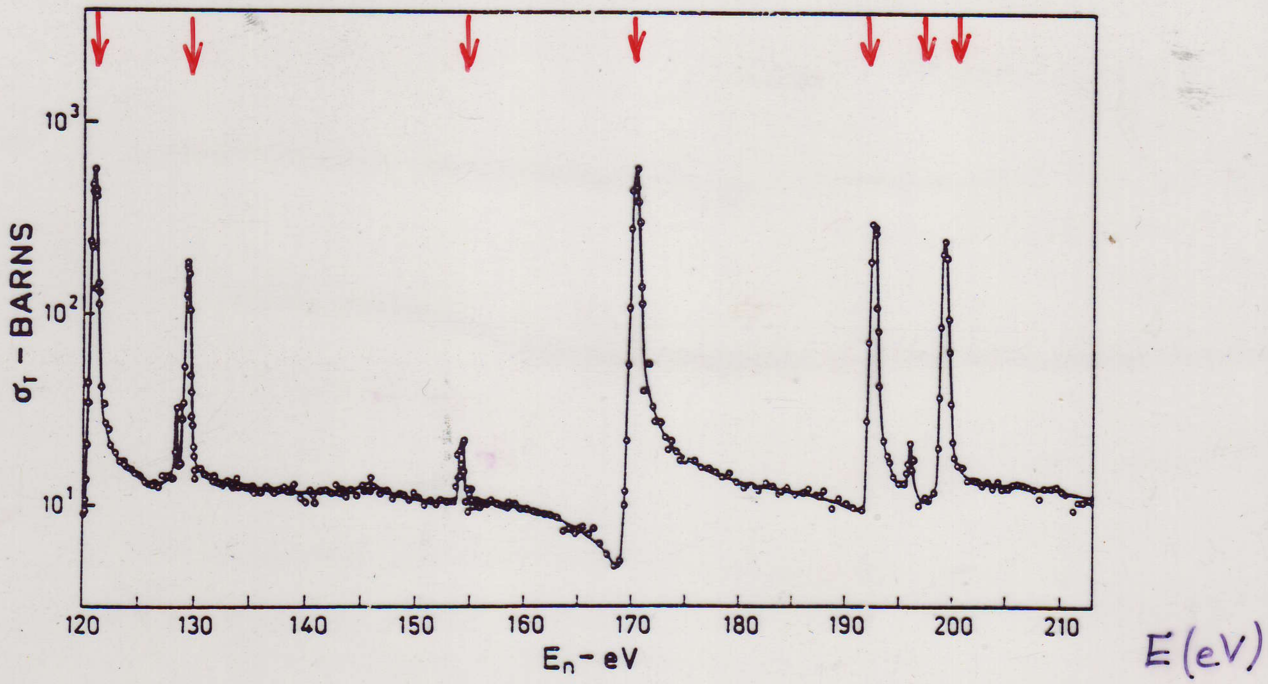


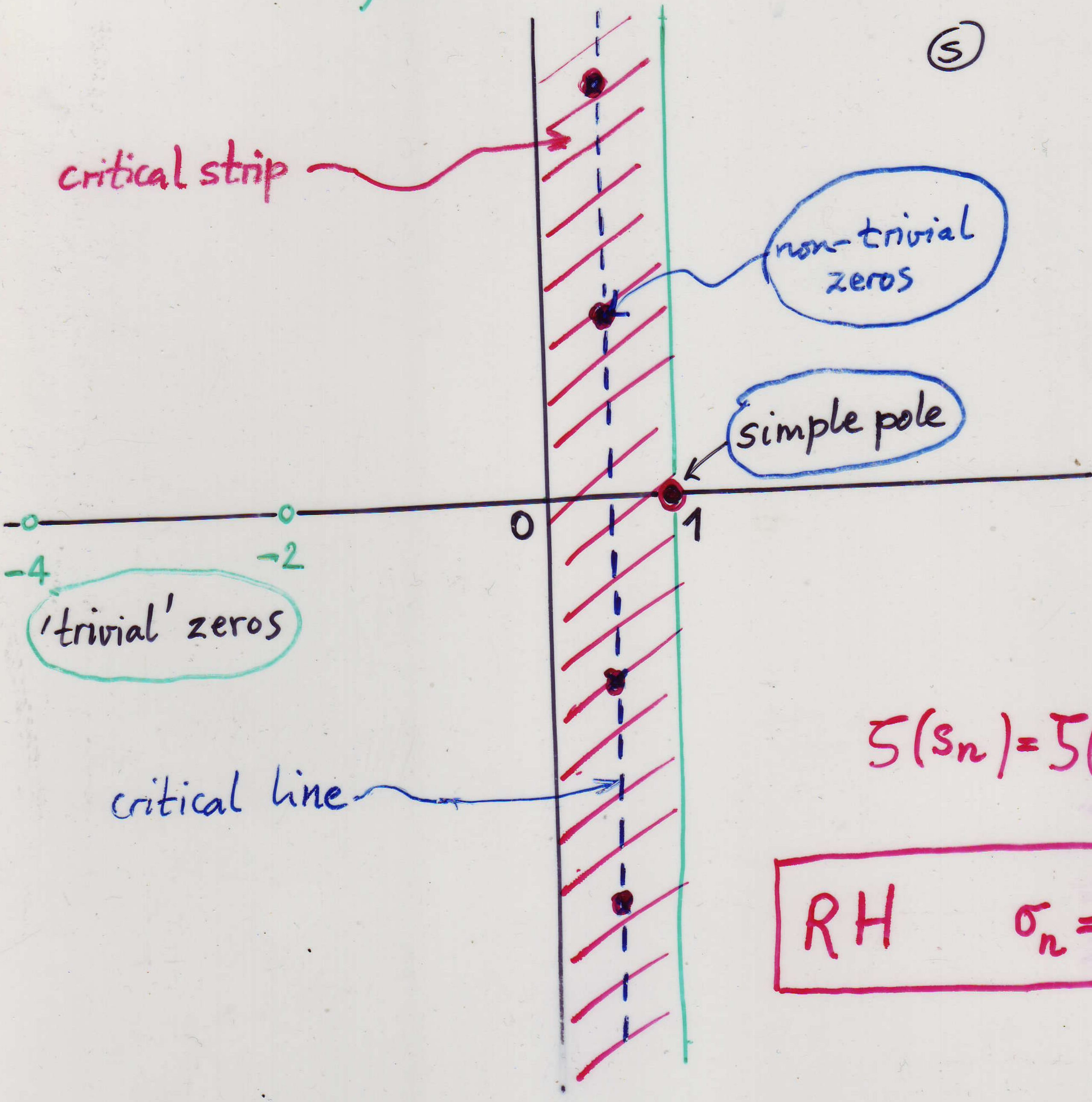
Fig. 3

Riemann's ζ-function

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

integers $\quad$ primes

$\operatorname{Re} s > 1$



$$\zeta(s_n) = \zeta(\sigma_n + it_n) = 0$$

$$\boxed{\text{RH} \qquad \sigma_n = \frac{1}{2}}$$

The Riemann Hypothesis.

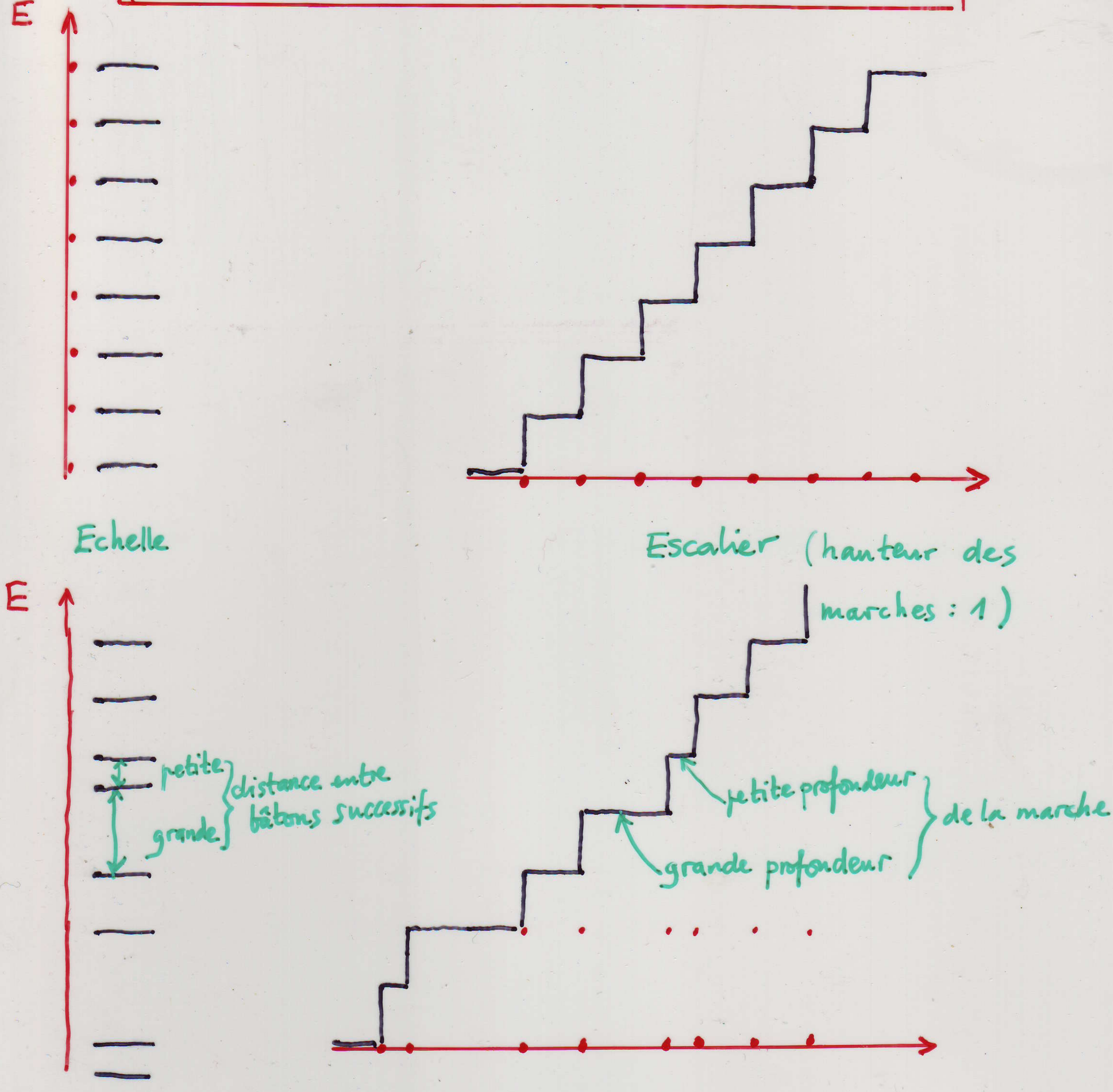
Bernhard Riemann, 'Über die Anzahl der Primzahlen unter einer gegebenen Größe' 1859

This function is finite for all finite values of t and can be developed as a power series in tt which converges very rapidly. Now since for values of s with real part greater than 1, $\log \zeta(s) = -\sum \log(1 - p^{-s})$ is finite and since the same is true of the other factors of $\xi(t)$, the function $\xi(t)$ can vanish only when the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. The number of roots of $\xi(t) = 0$ whose real parts lie between 0 and T is about

$$= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

because the integral $\int d \log \xi(t)$ taken in the positive sense around the domain consisting of all values whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T is (up to a fraction of the order of magnitude of $1/T$) equal to $[T \log (T/2\pi) - T]i$ and is, on the other hand, equal to the number of roots of $\xi(t) = 0$ in the domain multiplied by $2\pi i$. One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.

ECHELLES et ESCALIERS (SPECTRES)



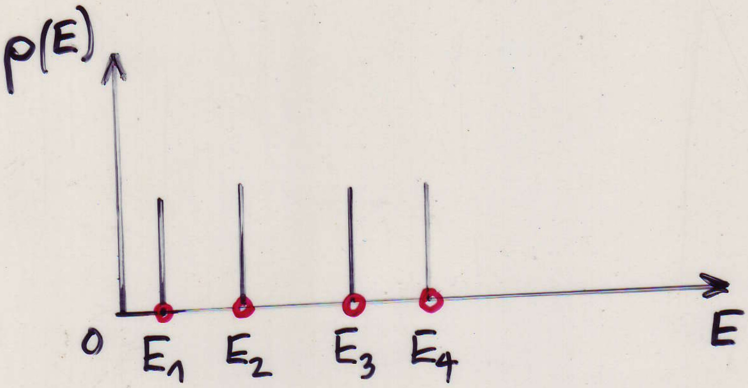
echelle : interdistance entre bâtons successifs

escalier : profondeurs des marches

rugosité (irregularité) des escaliers

density $\rho(E)=\sum_n \delta(E-E_n)$

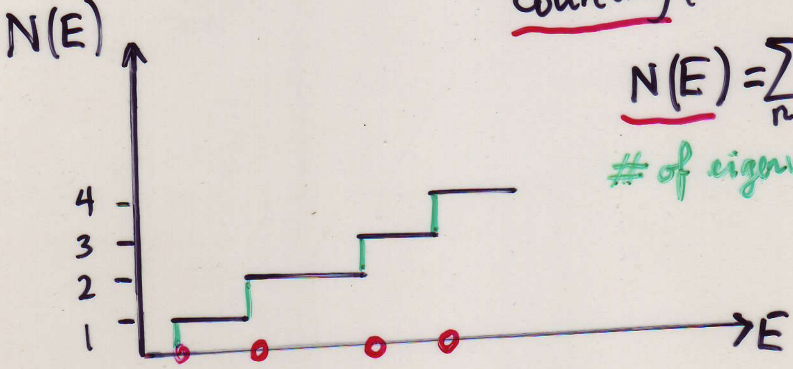
eigenvalues of the system



counting (staircase) function

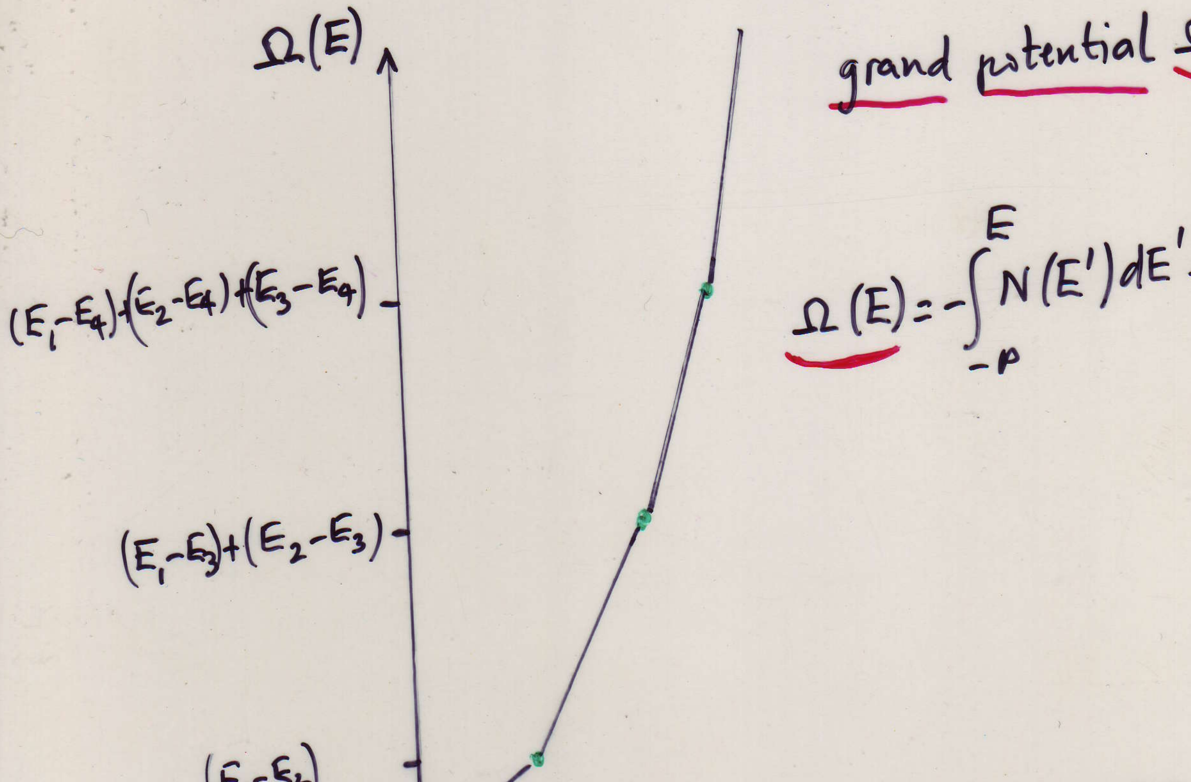
$$N(E)=\sum_n \theta(E-E_n)=\int_{-\infty}^{E}\rho(E')\,dE'$$

# of eigenvalues $\leq E$



grand potential $\Omega(E)$

$$\Omega(E)=-\int_{-p}^{E} N(E')\,dE'=\sum_n (E_n-E)\,\theta(E-E_n)$$



General strategy:

To decompose $O(E)$ in a smooth part $\bar{O}(E)$ plus the rest (oscillating part) $\tilde{O}(E)$

$$O(E) = \bar{O}(E) + \tilde{O}(E)$$

O may be the spectral density ρ

counting function N

grand potential Ω

Generally, the smooth part is the easy part and the oscillating / fluctuating part is the hard part

'The noise is the signal' (R. Landauer)

" ... the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix, which no particular properties except for its symmetric nature. "

Wigner, in *Symmetries and Reflections*

RANDOM MATRICES

$$\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & & & \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{pmatrix}$$

real symmetric $\beta=1$

hermitian $\beta=2$

real quaternion $\beta=4$

H_{ij} : r.i.v $(0,v^2)$ $\qquad \mathcal{P}(H) \propto \exp(-\mathrm{Tr}\, H^2)$

Eigenvalue distribution

$$\mathcal{P}(E_1, E_2, \ldots, E_N) \propto \exp\left\{-\frac{1}{4v^2}\sum E_i^2\right\} \prod_{i<j} |E_i - E_j|^\beta$$

Espectre

δ: espaiament entre dos valors propis successius, adjacents

densitat de probabilitat de δ

distribució de primers veïns

$$\boxed{p(s)}$$

Haq, Pandey & O.B.
Phys. Rev. Lett. 48 (1982) 1086

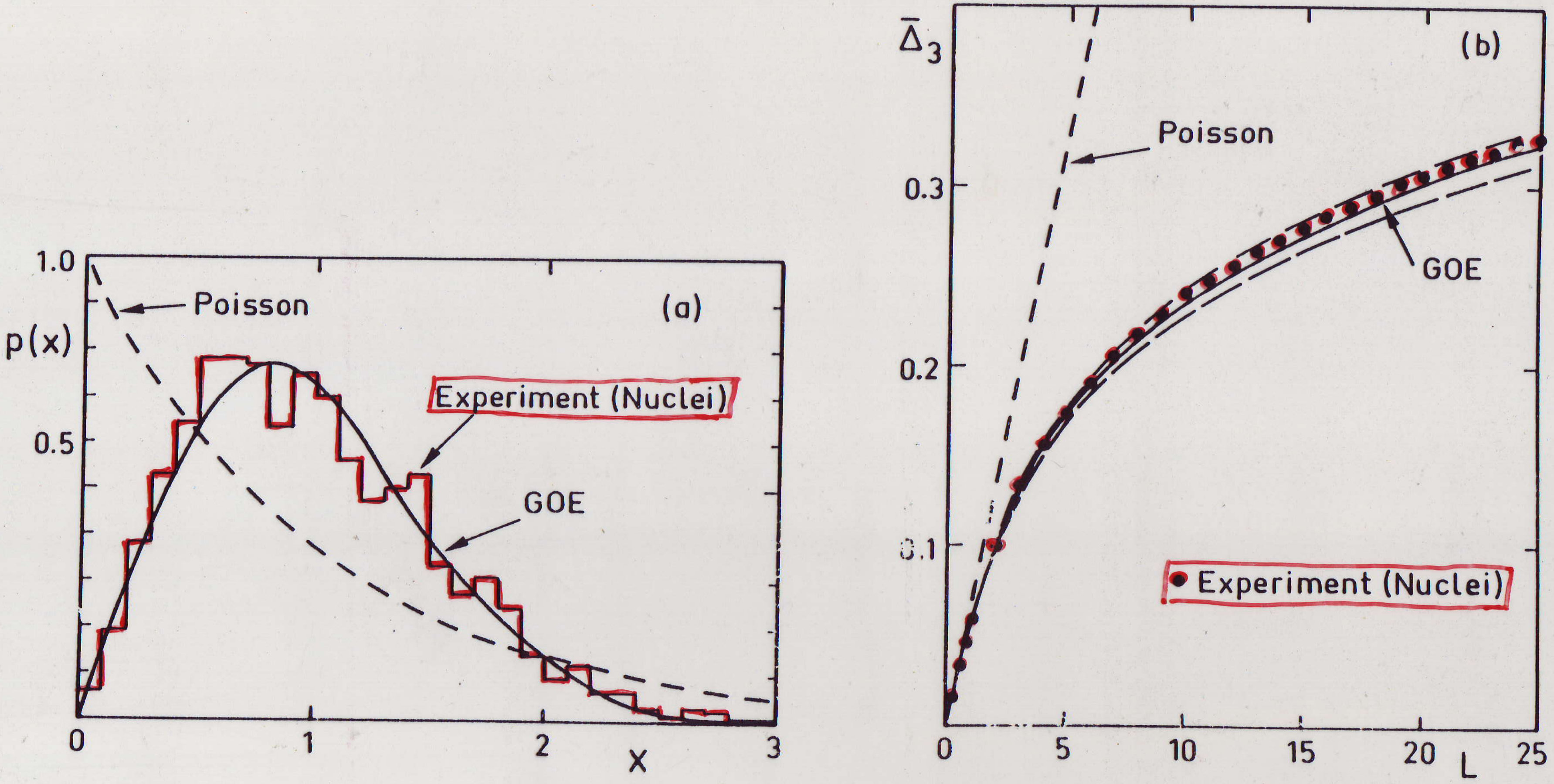


Fig. 8

smooth part

$$\begin{cases} \rho = \bar{\rho} + \tilde{\rho} \\ N = \bar{N} + \tilde{N} \\ \Omega = \bar{\Omega} + \tilde{\Omega} \end{cases}$$

~ oscillating part

$$\rho(E) = \sum_i \delta(E - t_i)$$

non-trivial zeros (on critical line)

smooth part

$$\begin{cases} \bar{\rho}(E) = \frac{1}{2\pi} \log \frac{E}{2\pi} + O(E^{-2}) \\ \bar{N}(E) = \frac{E}{2\pi} \log\left(\frac{E}{2\pi}\right) - \frac{E}{2\pi} + \frac{7}{8} + \frac{1}{48\pi E} + O(E^{-3}) \\ \bar{\Omega}(E) = -\frac{E^2}{4\pi} \log\left(\frac{E}{2\pi}\right) + \frac{3}{8\pi} E^2 - \frac{7}{8} E - \frac{\log E}{48\pi} + c + O(E^{-2}) \end{cases}$$

$$\boxed{\text{Riemann } \zeta\text{-function}}$$

Proc. London Math. Soc. 3 (1953) 99-117

SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

[Received 29 February 1952.—Read 20 March 1952]

Introduction

In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals. The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi . 63^2 < t < 2\pi . 64^2$ was investigated during that period, and very little more was accomplished.

The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed. The procedure was such that if it had been accurately followed, and if the machine made no errors in the period, then one could be sure that there were no zeros off the critical line in the interval in question. In practice only a few of the results were checked by repeating the calculation, so that the machine might well have made an error.

If more time had been available it was intended to do some more calculations in an altogether different spirit. There is no reason in principle why computation should not be carried through with the rigour usual in mathematical analysis. If definite rules are laid down as to how the computation is to be done one can predict bounds for the errors throughout. When the computations are done by hand there are serious practical difficulties about this. The computer will probably have his own ideas as to how certain steps should be done. When certain steps may be omitted without serious loss of accuracy he will wish to do so. Furthermore he will probably not see the point of the 'rigorous' computation and will probably say 'If you want more certainty about the accuracy why not just take more figures?' an argument difficult to counter. However, if the calculations are being done by an automatic computer one can feel sure that this kind of indiscipline does not occur. Even with the automatic computer this rigour can be rather tiresome to achieve, but in connexion with such a subject as the analytical theory of numbers, where rigour is the essence, it seems worth while. Unfortunately, although the details were all worked out, practically nothing was done on these lines. The interval $1414 < t < 1608$ was investigated and checked, but unfortunately at this point the machine broke down and no further work was done. Furthermore this interval was subsequently found to have been run with a wrong error value, and the most that can consequently be asserted with certainty is that the zeros lie on the critical line up to $t = 1540$, Titchmarsh having investigated as far as 1468 (Titchmarsh (**5**)).

This paper is divided into two parts. The first part is devoted to the analysis connected with the problem. All the results obtained in this part are likely to be applicable to any further calculations to the same end, whether carried out on the Manchester Computer or by any other means. The second part is concerned with the means by which the results were achieved on the Manchester Computer.

The principal investigation concerned the range $63^2 \leqslant \tau \leqslant 64^2$, i.e. the interval in which $m = 63$. Working at full efficiency it should have taken about 4 hours to calculate these values, the number of zeros concerned being about 1070. Full efficiency was not, however, achieved, and the calculation took about 9 hours. Only a small amount of this additional time was accounted for by duplicating the work. The special investigations in the neighbourhood of points where the unexpected sign occurred took a further 8 hours. The general reliability of the machine was checked from time to time by repeating small sections. The recorded cumulants were useful in this connexion. These cumulants were the totals of the values of $Z(\tau)$ computed since the last recorded value. If a calculation is repeated and there is agreement in cumulant value then there is a strong presumption that there is also agreement in all the individual values contributing to it. The result of this investigation, so far as it can be relied on, was that there are no complex zeros or multiple real zeros of $Z(\tau)$ in the region

$$63^2 \leqslant \tau \leqslant 64^2,$$

i.e. all zeros of $\zeta(s)$ in the region $2\pi . 63^2 \leqslant t \leqslant 2\pi . 64^2$ are simple zeros on the critical line.

Another investigation was also started with a view to extending the range of relatively small values of t for which the Riemann hypothesis holds. Titchmarsh has already proved that it holds up to $t = 1468$, i.e. to about $\tau = 231$. The new investigation started somewhat before $\tau = 225$ to allow a margin for the application of Theorem 5. It was intended to continue the work up to about $\tau = 500$, but an early breakdown resulted in its abandonment at $\tau = 256$. After applying Theorem 5 it would only be possible to assert the validity of the Riemann hypothesis up to about $\tau = 250$. All this part of the calculations was done twice, the unrecorded values being confirmed by means of the 'cumulants'.

Unfortunately $0{\cdot}31E$ was given the inappropriate value of $\frac{1}{128}$ and consequently we are only able to assert the validity of the Riemann hypothesis as far as $t = 1540$, a negligible advance.

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2. Jahnke u. Emde (1), *Tafeln höherer Funktionen*, Leipzig, 1948.
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4. —— 'The zeros of the Riemann zeta-function', *Proc. Roy. Soc.* A, 151 (1935), 234–55.
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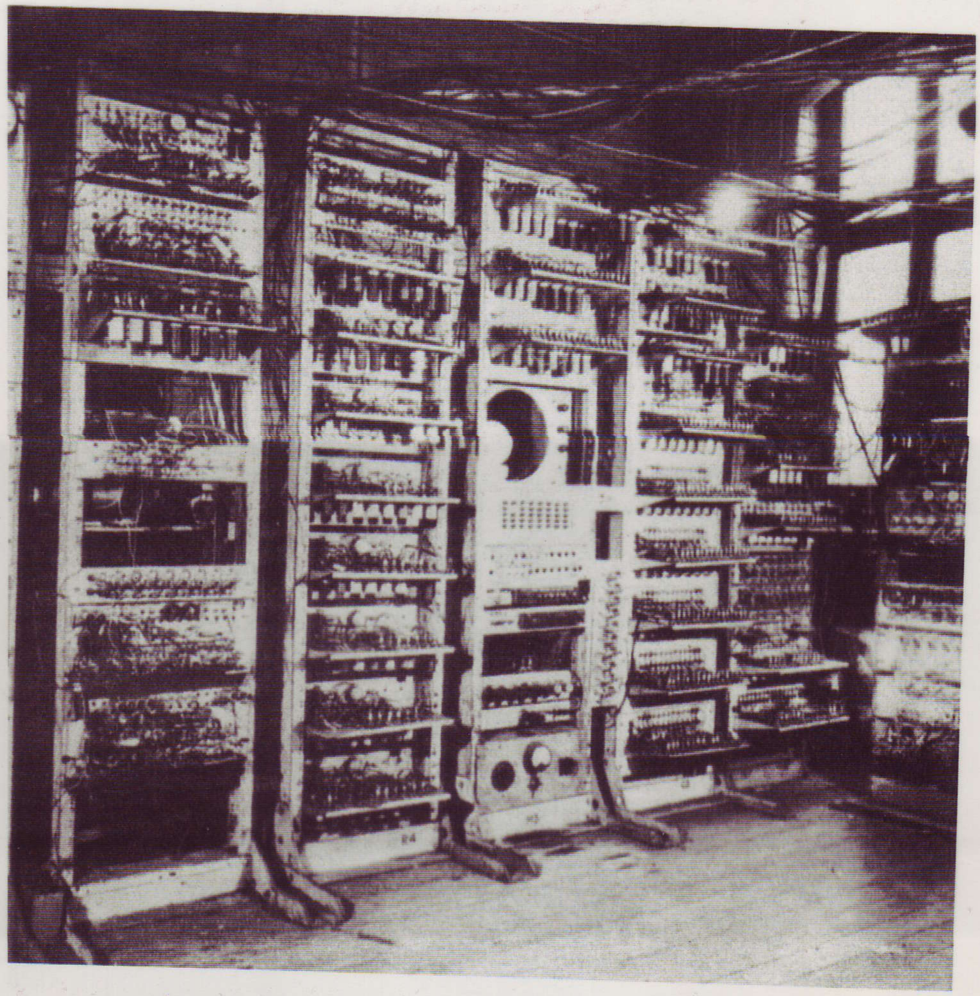


The prototype Manchester computer

$$Z(t) = \zeta\left(\tfrac{1}{2} + it\right) e^{i\vartheta(t)}$$

$$\vartheta(t) = \arg \Gamma\left(\tfrac{1}{4} + i\tfrac{t}{2}\right) - \frac{\log \pi}{2} t \qquad t \geq 0$$

For t real, $Z(t)$ real

$$|Z(t)| = \left|\zeta\left(\tfrac{1}{2} + it\right)\right|$$

Counting sign changes of $Z(t)$ $(t \geq 0)$ provides

\# zeros on critical line $\quad 0 \leq t \leq T$

$\frac{1}{\pi} \operatorname{Re}\left[\vartheta(T) - i \log \zeta\left(\tfrac{1}{2} + iT\right)\right] + 1$ provides

\# zeros on critical strip

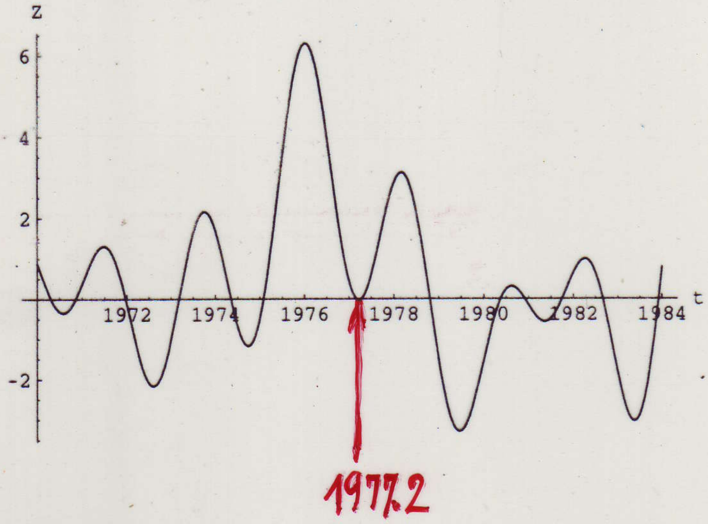
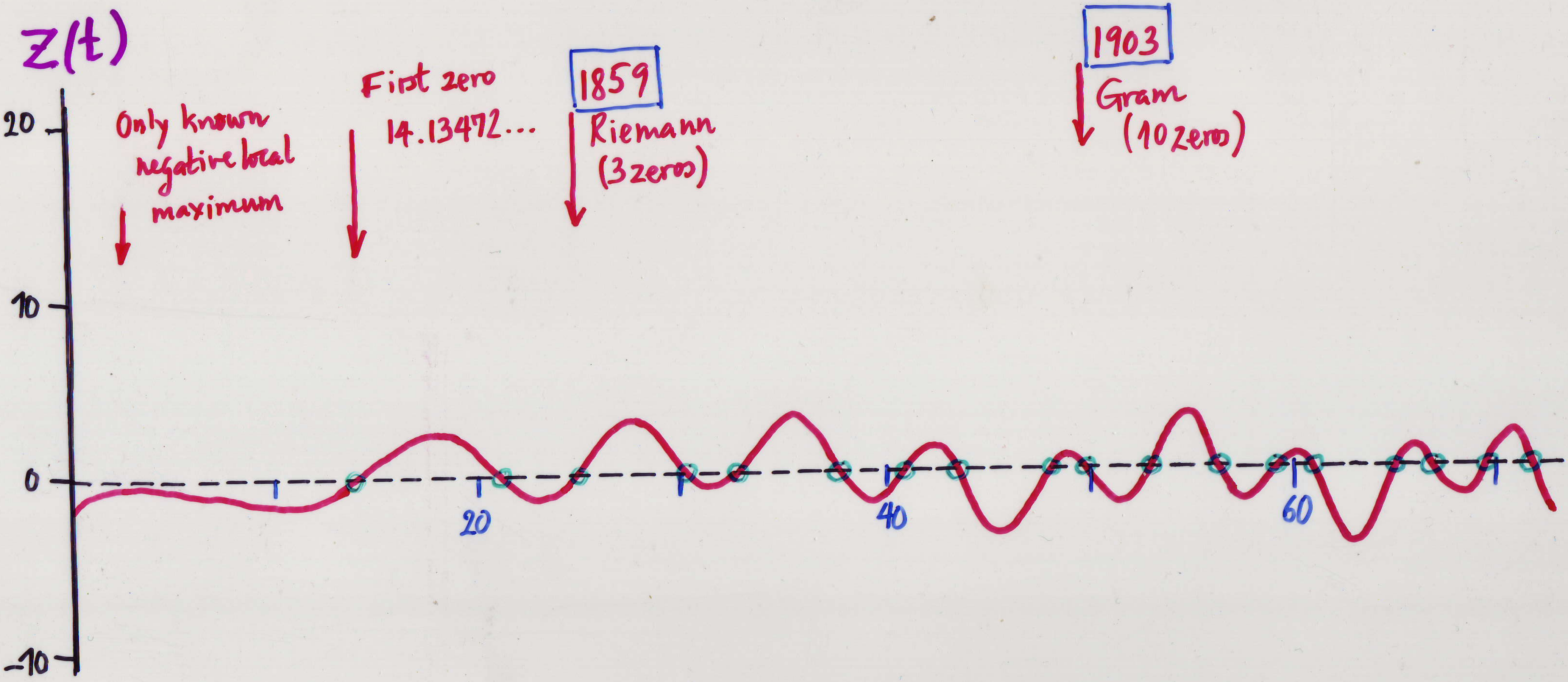


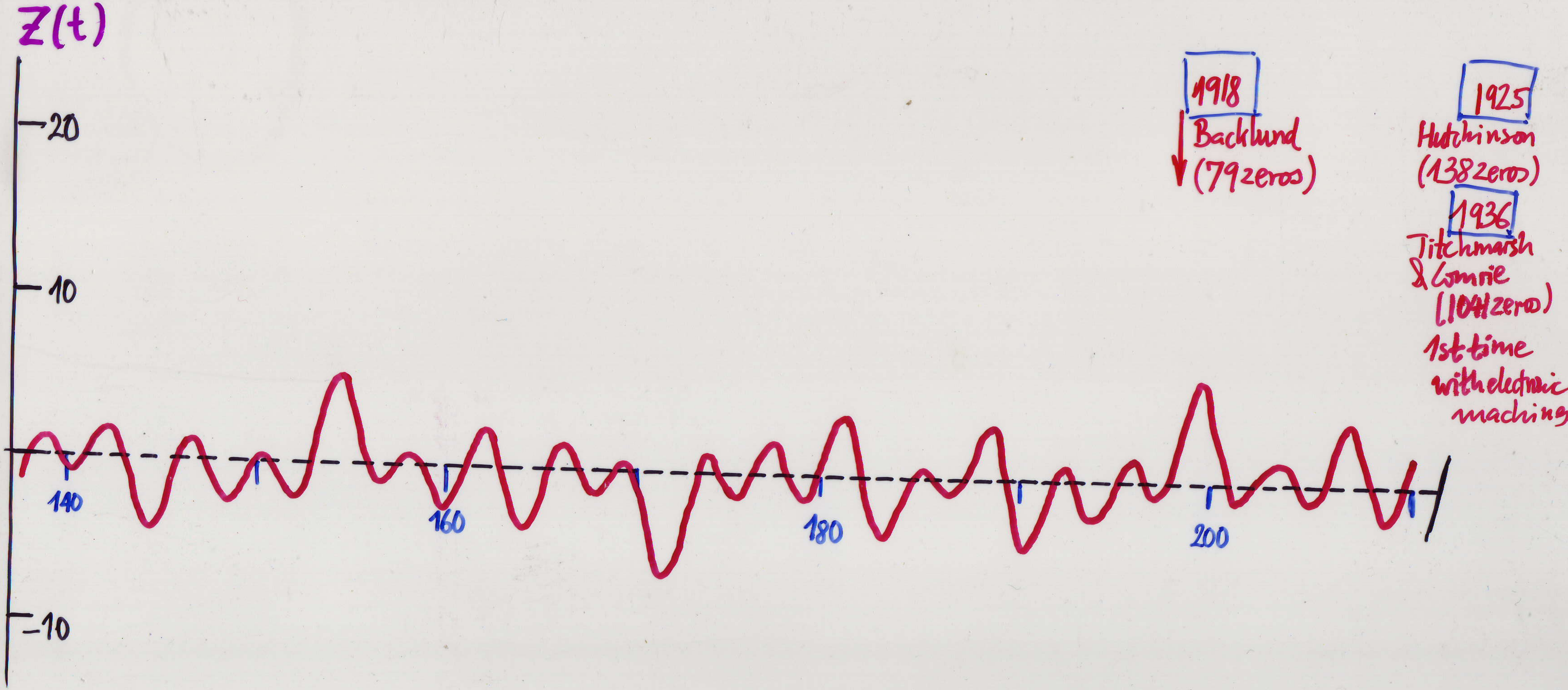
FIG. 6. The Hardy function $\mathcal{Z}(t)$, Eq.(refhardy) showing two close-lying zeros of ζ around $t = 7702$.

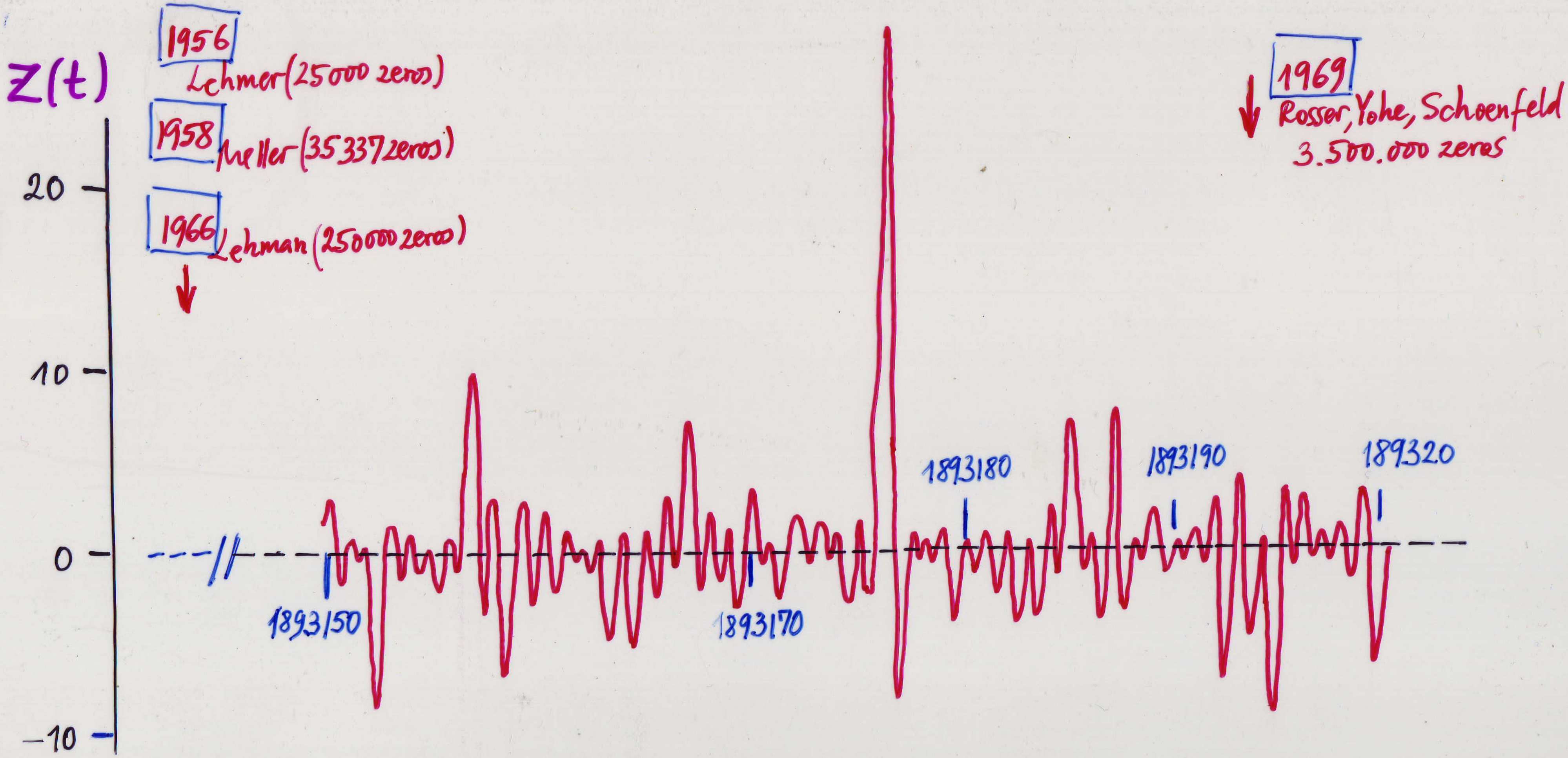
$$Z(t) = \exp\left(i\theta(t)\right)\zeta\left(\tfrac{1}{2}+it\right)$$

$$\theta(t) = \mathrm{Im}\log\Gamma\left(\tfrac{1}{4}+i\tfrac{t}{2}\right)-(t\log\pi)/2$$

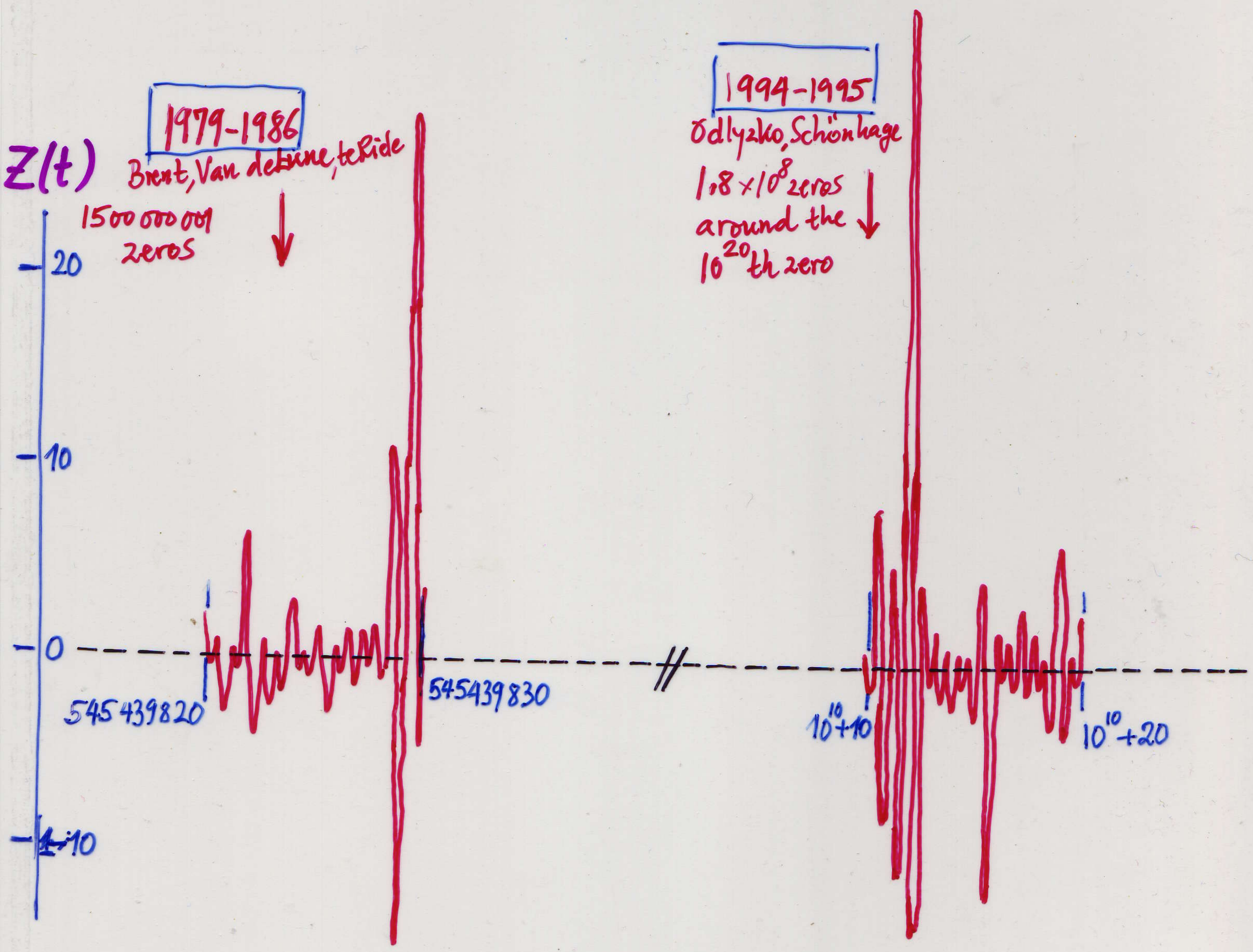


Z(t)
20
10
0
-10
Only known negative local maximum
First zero 14.13472...
1859
Riemann (3 zeros)
1903
Gram (10 zeros)
20
40
60


Z(t)
20
10
-10
140
160
180
200
1918
Backlund
(79 zeros)
1925
Hutchinson
(138 zeros)
1936
Titchmarsh
& Comrie
(1041 zeros)
1st time
with electronic
machines



Z(t)
1956
Lehmer (25000 zeros)
1958
Meller (35337 zeros)
1966
Lehman (250000 zeros)
1969
Rosser, Yohe, Schoenfeld
3.500.000 zeros
20
10
0
-10
1893150
1893170
1893180
1893190
189320



Z(t)
1979-1986
Brent, Van de Lune, te Riele
1500000001
zeros
20
10
0
-10
545439820
545439830
1994-1995
Odlyzko, Schönhage
1,8 × 10^8 zeros
around the
10^20 th zero
10^10+10
10^10+20

$\zeta\left(\frac{1}{2}+it\right)=0$

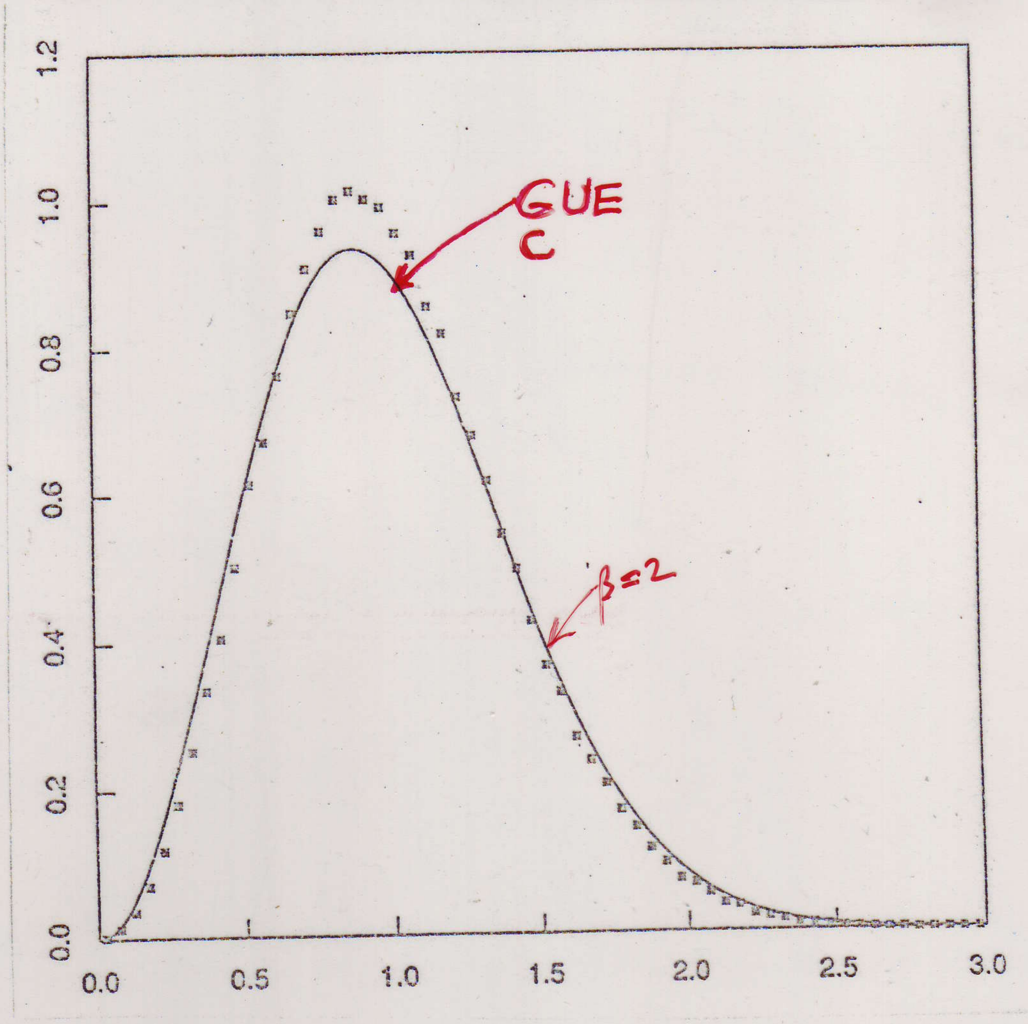
$\sim 10^{22}$ th zero

$t = 1\,370\,919\,909\,931\,995\,308\,226.62751\ldots$

A. Odlyzko '99

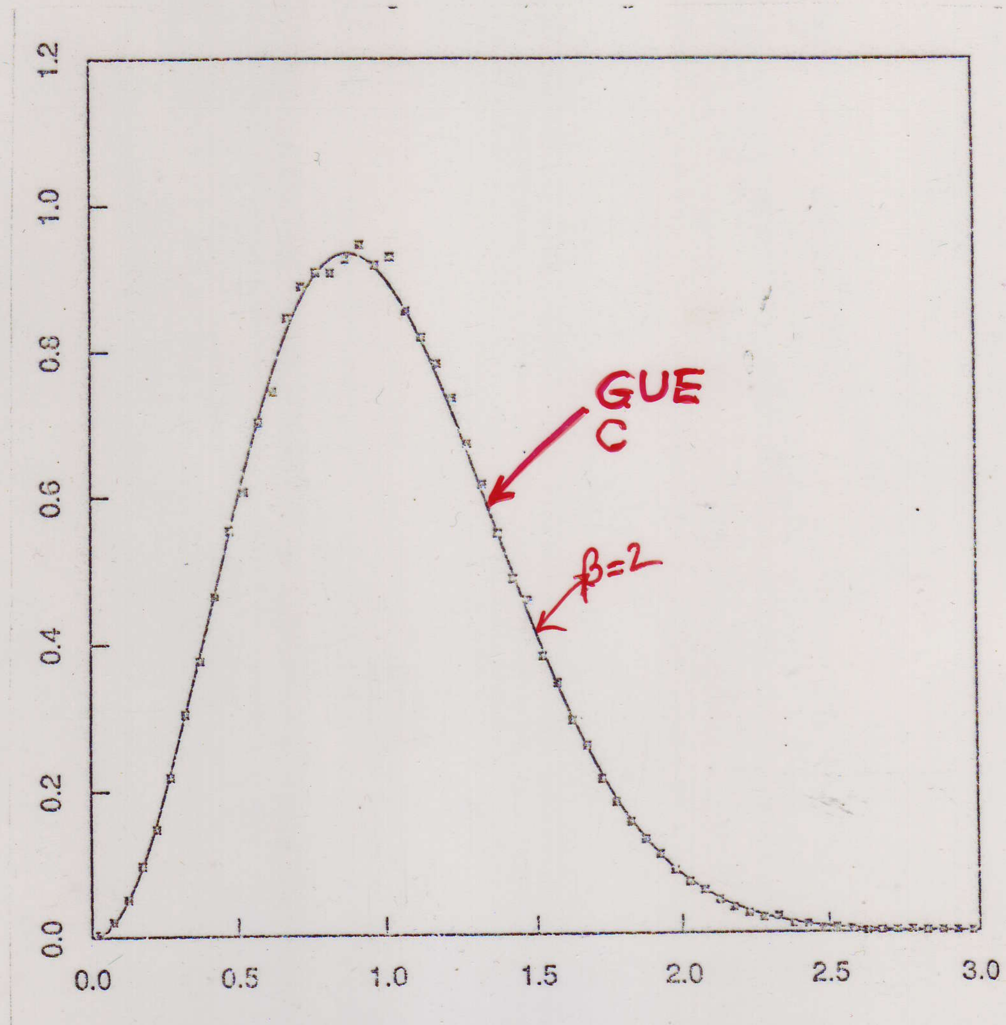
$10^{22} - 61110 \leq n \leq 10^{22} + 5311\,191\,361$

spacing distribution



10^5 zeros

N=0



10^5 zeros

$N=10^{12}$

zeros from $\zeta(s)$ From A. Odlyzko ('80's)

$$\boxed{\text{Montgomery-Odyzko conjecture}}$$

Riemann's zeros, high on the critical line, behave statistically as eigenvalues of hermitean (or unitary) random matrices

c.f. Montgomery,

2-point correlation function

$$\boxed{Y_2(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \quad \text{like RMT for } \beta=2!}$$

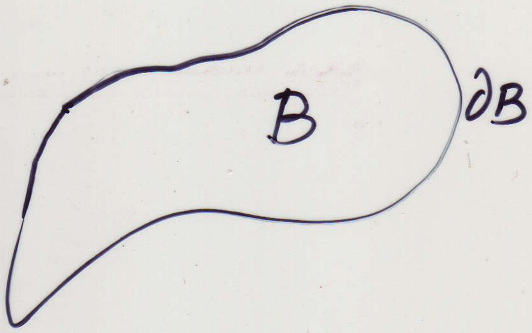
1972

Clàssic

billar

Quàntic

partícula lliure en una capsa



Modes propis de la capsa: solucions de equació de Schrödinger (o de Helmholtz) $(\Delta + E)\Psi = 0$

$\psi\Big|_{\text{frontera}} = 0$ Dirichlet

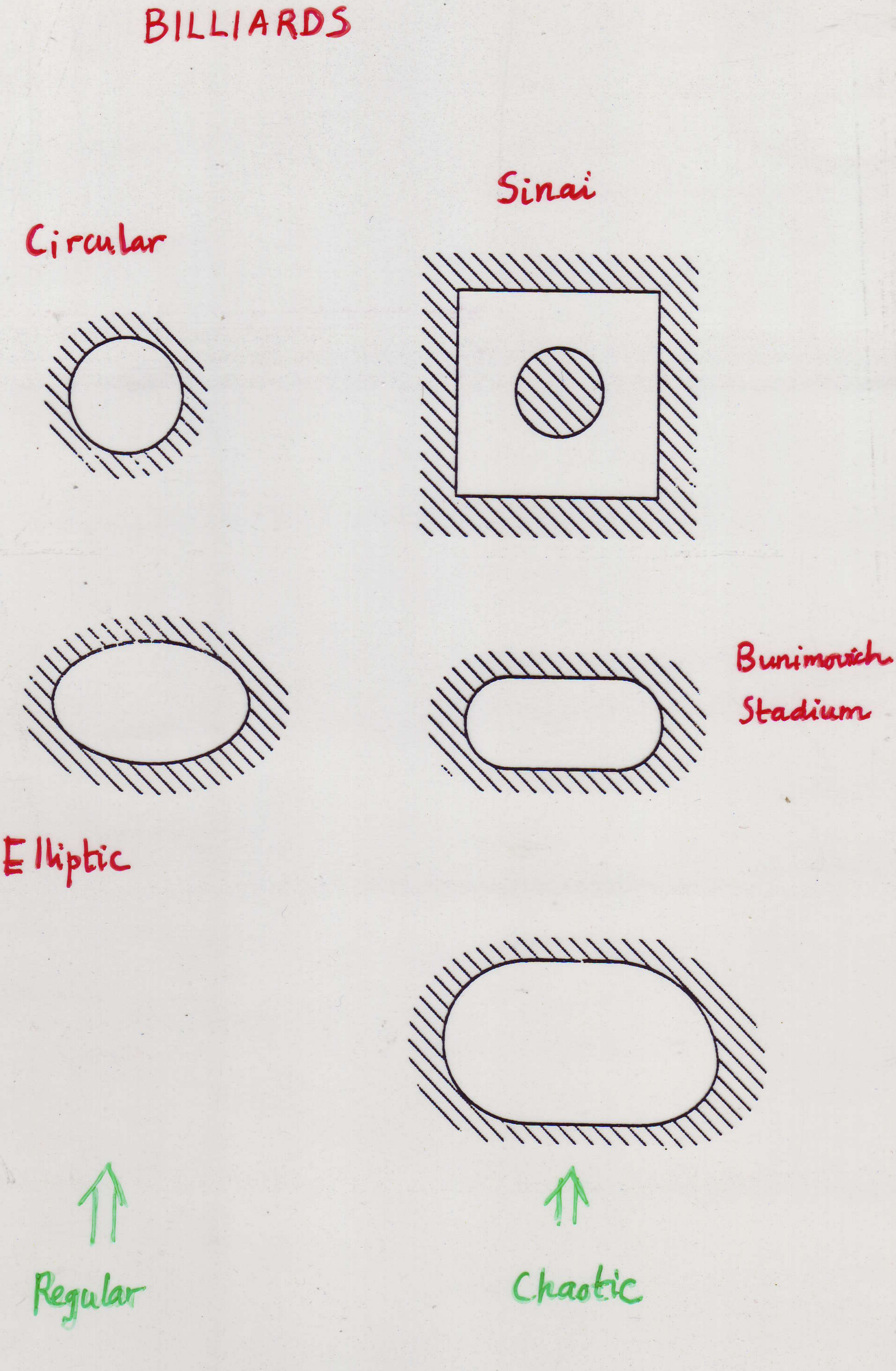
Espectre del laplacià

$\dfrac{d\psi}{dn}\Big|_{\text{frontera}} = 0$ Neuman

$N(E)$: funció de comptatge (escala)

Weyl $\bar{N}(E) = \frac{1}{4\pi}\left(\underbrace{S}_{\text{superfície}} E \mp \underbrace{\alpha}_{\text{perímetre}}\sqrt{E} + \underbrace{K}_{\text{integral de curvatura}} + \dots\right)$

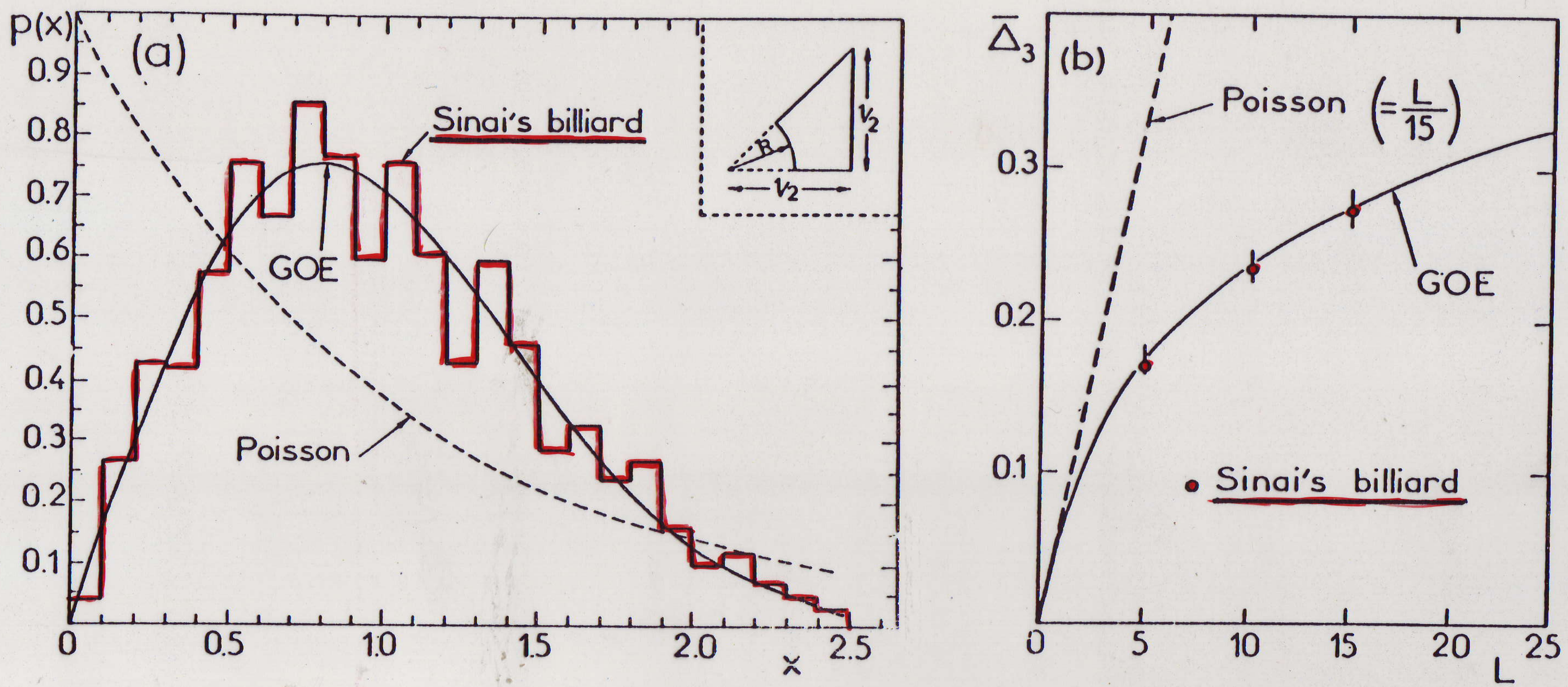
M. Kac: 'Can one hear the shape of a drum'?



BILLIARDS
Circular
Sinai
Elliptic
Bunimovich Stadium
Regular
Chaotic

Fig. 11

Chaotic system Sinai billiard



Bohigas, Giannoni & Schmit
Phys. Rev. Lett. 52 (1984) 1

Fig. 12

Hermitian random matrices can serve as models of Hamiltonians of chaotic autonomous systems (cf GOE, GUE, GSE).

Unitary random matrices (COE, CUE, CSE) play an analogous role for chaotic periodically driven systems, as models of the unitary evolution operator.

$\boxed{\text{BGS or random matrix conjecture}}$

In contrast, spectral fluctuations of regular systems are Poissonian (Berry & Tabor).
Sinai

$\boxed{\text{Poincaré}}$: partir des solutions périodiques (les corps mobiles décrivent des courbes fermées) ;

étudier ce qu'il se passe au voisinage d'une trajectoire périodique

'(l'étude des trajectoires périodiques) est la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.'

A Scandinavian episode (before NORDITA!)

Oscar II, King of Sweden. Graduated from Uppsala

Gösta Mittag-Leffler: graduated also from Uppsala (1872)

Went as a 'postdoc' to Paris (Hermite) and Berlin (Weierstraß).

Came back to Sweden and became professor at Stockholm.

Convinced Oscar II to publish Acta Mathematica and later to create a prize

subjects selected by Hermite and Weierstraß, (joined by Mittag-Leffler as jury members)

Deadline for submission: June 1888

Should be awarded January 89, Oscar's II birthday

'Sur le problème à trois corps et les équations de la dynamique'.

Edward Phragmén, July 89

Final version: end of 1890

The density of states may be computed from the Green's function

$$\rho(E) = -\frac{1}{\pi}\int d\underset{\sim}{q} \lim_{\epsilon \to 0^+} G(\underset{\sim}{q}, \underset{\sim}{q}, E+i\epsilon)$$

$G(q, q, E)$ is the propagator for paths of energy E that start at $\underset{\sim}{q}$ and come back to $\underset{\sim}{q}$. In the limit $\hbar \to 0$, the leading contribution to G is a sum over all classical trajectories that start and come back to $\underset{\sim}{q}$. A stationary phase approximation of the integral over $\underset{\sim}{q}$ selects, among all the classical closed trajectories, those that start and come back to a given point with the same momentum (periodic trajectories).

Periodic orbit theory. cf. Gutzwiller, Bloch & Balian

$$\rho(E) = \sum \delta(E - E_i)$$

Based on the saddle-point approximation to Feynman's path integral for $\hbar \to 0$

$$\rho(E) = \langle \rho(E) \rangle + \rho_{fe}(E)$$

$$\rho_{fe}(E) \simeq \sum_{p.o} A e^{iS}$$

$$\underbrace{\sum \delta(E - E_i)}_{\text{Quantum}} \simeq \sum_{p.o.} \underbrace{A e^{iS}}_{\text{Classical information}}$$

Interval ΔE	energy scales	Corresponds to microscope	time scales	# of levels it contains
(1) h^d	mean spacing δ	h^{-d}	Heisenberg time $\tau_H = \frac{h}{\delta}$	1
(2) h	h/τ_{min}	h^{-1}	period of shortest periodic orbit τ_{min}	$h^{-(d-1)}$

h : Planck's constant

d : # of freedoms

(1) Inner scale : Dominated by properties of extremely long orbits

(2) Outer scale : The orbits which are not long differ from system to system and break down universalities at large scale. The outer scale is classically small but semiclassically large.

Proc. Roy. Soc. London A400 (1985) 229

M.V. Berry

SEMICLASSICAL THEORY

Gutzwiller
Balian & Bloch
Berry

$$\rho(E) = \sum_i \delta(E - E_i)$$

$$\rho(E) = \underbrace{\bar{\rho}(E)}_{\text{smooth}} + \underbrace{\tilde{\rho}(E)}_{\text{oscillating}}$$

Gutzwiller trace formula

$$\tilde{\rho}(E) = 2 \sum_{p} \sum_{r=1}^{\infty} A_{p,r}(E) \cos\left[r S_p(E)/\hbar + \nu_{p,r} \right]$$

quantum mechanics

primitive periodic orbits

repetitions

classical mechanics

S_p action ; $\tau_p = dS_p/dE$ period of periodic orbit p

$\nu_{p,r}$ Maslov index

$$A_{p,r} = \frac{\tau_p}{h\sqrt{|\det[M_p^r - I]|}}$$

Chaotic systems

$$A_p^2 = \frac{(2\pi)^{d-1}}{\hbar^{(d+1)}\,\tau_p^d\,\left|\det\{\partial\omega_j/\partial I_k\}_p \sum_j \omega_j \cdot \frac{\partial I_j}{\partial \tau_p}\right|}$$

Integrable systems

d : # degrees of freedom

h : Planck's constant

(I, ω) : action-angle variables

frequencies $\omega_j = \dfrac{\partial H}{\partial I_j} = 2\pi \dfrac{m_j}{\tau_p}$

m_j : integers

$\boxed{\text{Integrable systems}}$

action I_j, angle ϑ_j variables

caracteristic frequencies $\omega_j = \dfrac{\partial H}{\partial I_j}$

quantization

$$\omega_j(I_1, \ldots, I_d) = 2\pi\, m_j / \tau$$

commensurability of frequencies

m_j : integers

$$A_p^2(E) = \frac{(2\pi)^{d-1}}{\hbar^{d+1}\, \tau_p^d \left| \det\left\{\dfrac{\partial \omega_i}{\partial I_k}\right\}_p \sum_j \omega_j \cdot \dfrac{\partial I_j}{\partial \tau_p} \right|}$$

For $\boxed{\text{chaotic systems}}$

$$A_{p,r}(E) = \frac{\tau_p}{2\pi\hbar\sqrt{\left|\det\left(M_p^r - I\right)\right|}}$$

M_p : monodromy matrix

governs the motion in the vicinity of the periodic motion. Contains the information on the unstability

When there is a Lyapunov exponent much larger than the others

$$A_{p,r}(E) \approx \frac{\tau_p}{2\pi\hbar} \exp\left(-\frac{r}{2}\lambda_p^{max}\right)$$

SEMICLASSICS

$$K_{GOE}(\tau) = 2\tau - \tau \log(1+2\tau) \qquad 0 \leq \tau \leq 1$$

$$= 2\tau - 2\tau^2 + O(\tau^3)$$

leading off-diagonal correction

M. Sieber, K. Richter, Physica Scripta T90:128, 2001

M. Sieber, J. Phys. A35 (2002) L613

free motion on a Riemann surface of constant negative curvature

also Turek, Richter, preprint

Berkolaiko, Schanz, Whitney, Phys. Rev. Lett. 88 (2002) 104101

for generic quantum graphs

B.S.W., nlin.CD/0205014

$$K(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + O(\tau^4)$$

for a family of quantum graphs

D. Spehner, nlin.CD/0303051

extends Sieber, Richter for general 2-dimensional hyperbolic systems

S. Müller, S. Heusler, P. Braun, F. Haake, A. Altland

'Semiclassical foundation of universality in Quantum Chaos'

nlin/CD/0401021, Phys. Rev. Lett. 93 (2004) 014103-1-4

Phys. Rev. E72 (2005) 046207

ζ - function

$$\tilde{\rho}(E) = -\frac{1}{\pi} \sum_{p} \sum_{r=1}^{\infty} \frac{\log p}{p^{r/2}} \cos(Er \log p)$$

'quantum'

primes

repetitions

'classical'

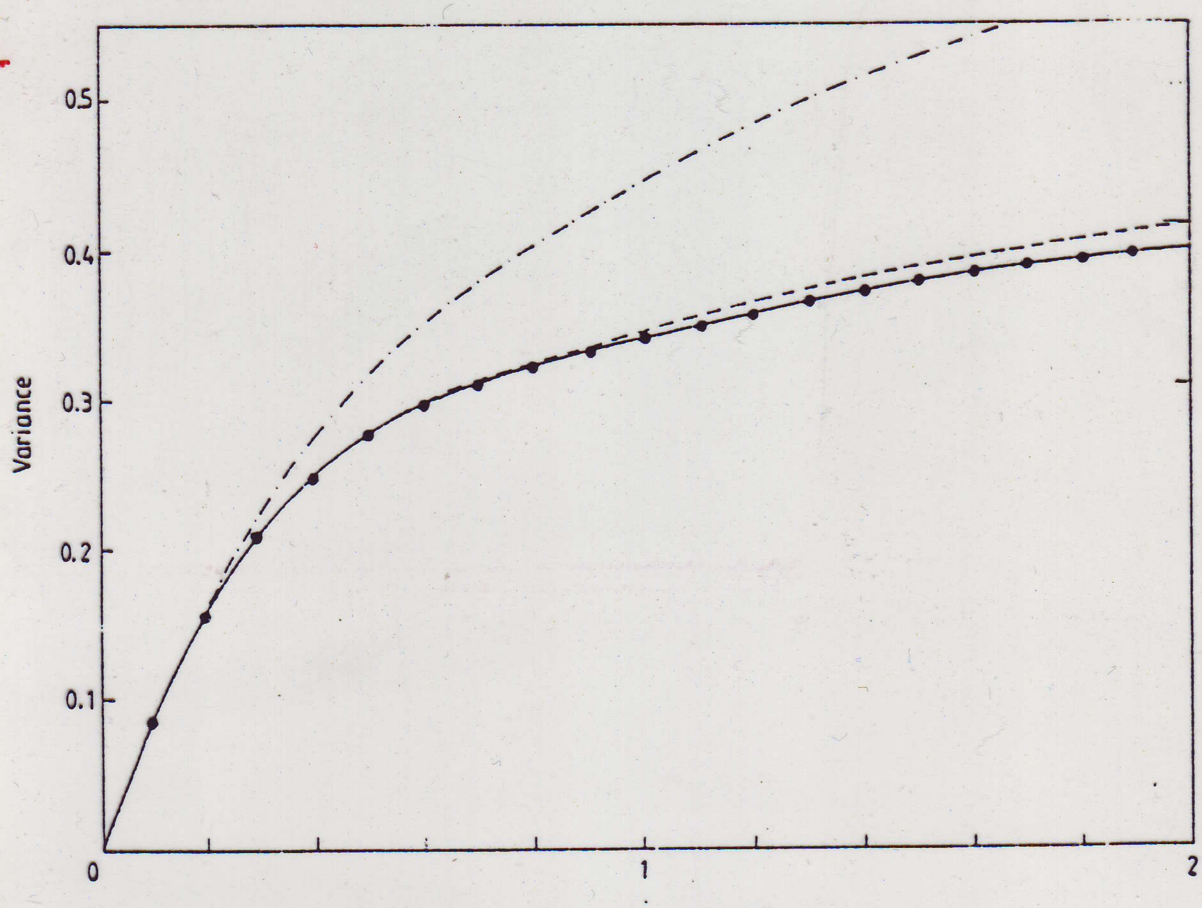
Dictionnary

primitive periodic orbits	$\longrightarrow$	prime numbers p
repetitions	$\longrightarrow$	r
Planck constant $\hbar$	$\longrightarrow$	1
action S_p	$\longrightarrow$	$E \log p$
period τ_p	$\longrightarrow$	$\log p$
$\lvert\det(M_p^r - I)\rvert$	$\longrightarrow$	p^r
amplitude $A_{p,r}$	$\longrightarrow$	$\log p / (2\pi p^{r/2})$
Maslov index $\nu_{p,r}$	$\longrightarrow$	π

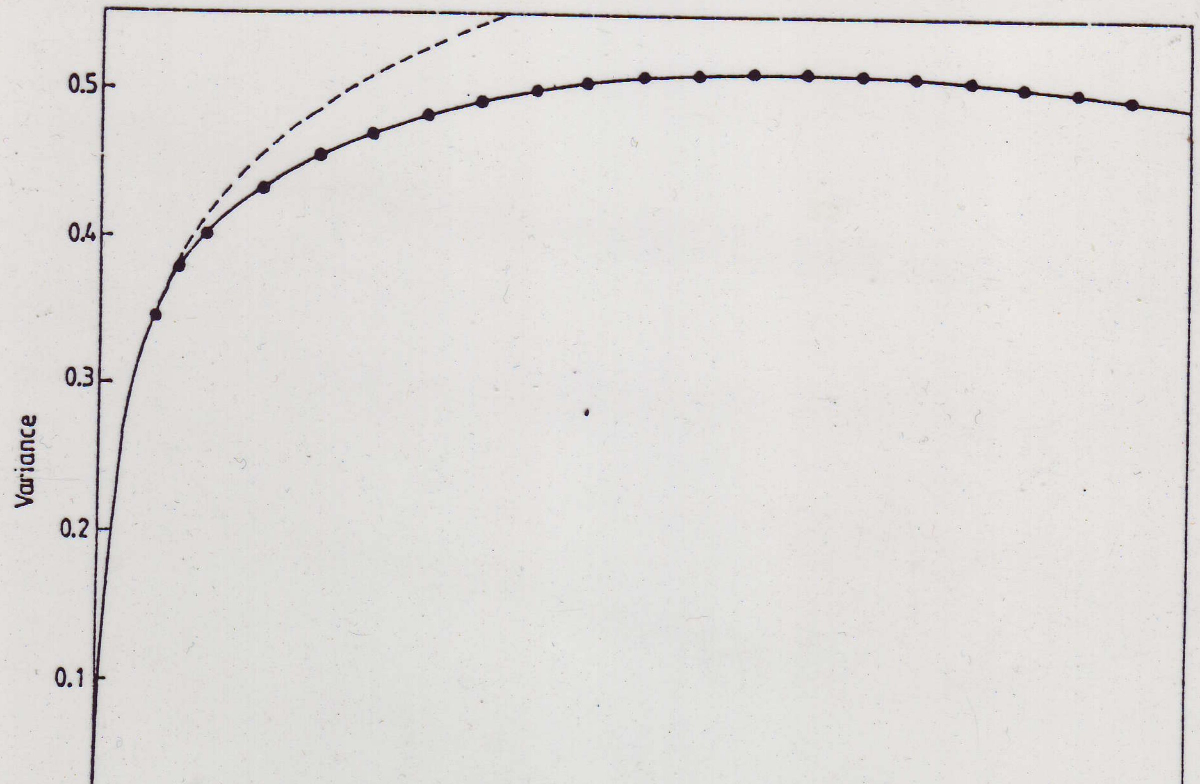
Berry, Keating, ...

ζ(s) : number variance   zeros from Odlyzko

Σ^2



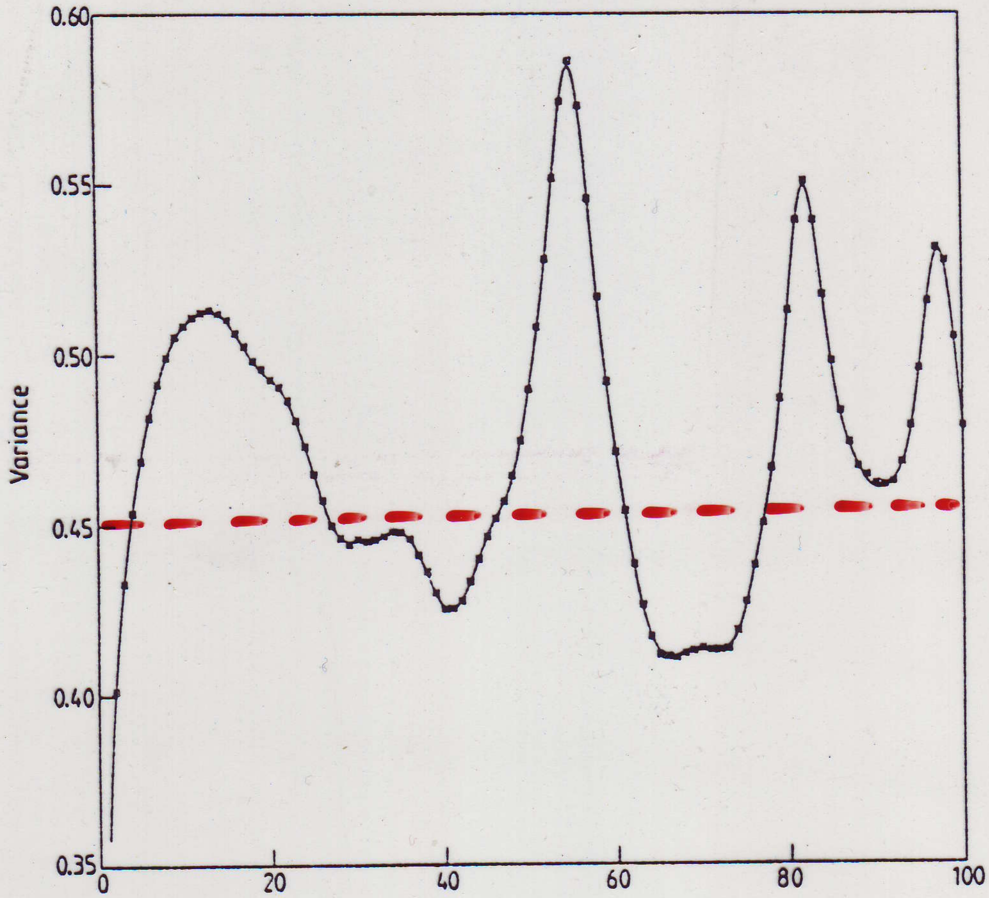
Σ^2



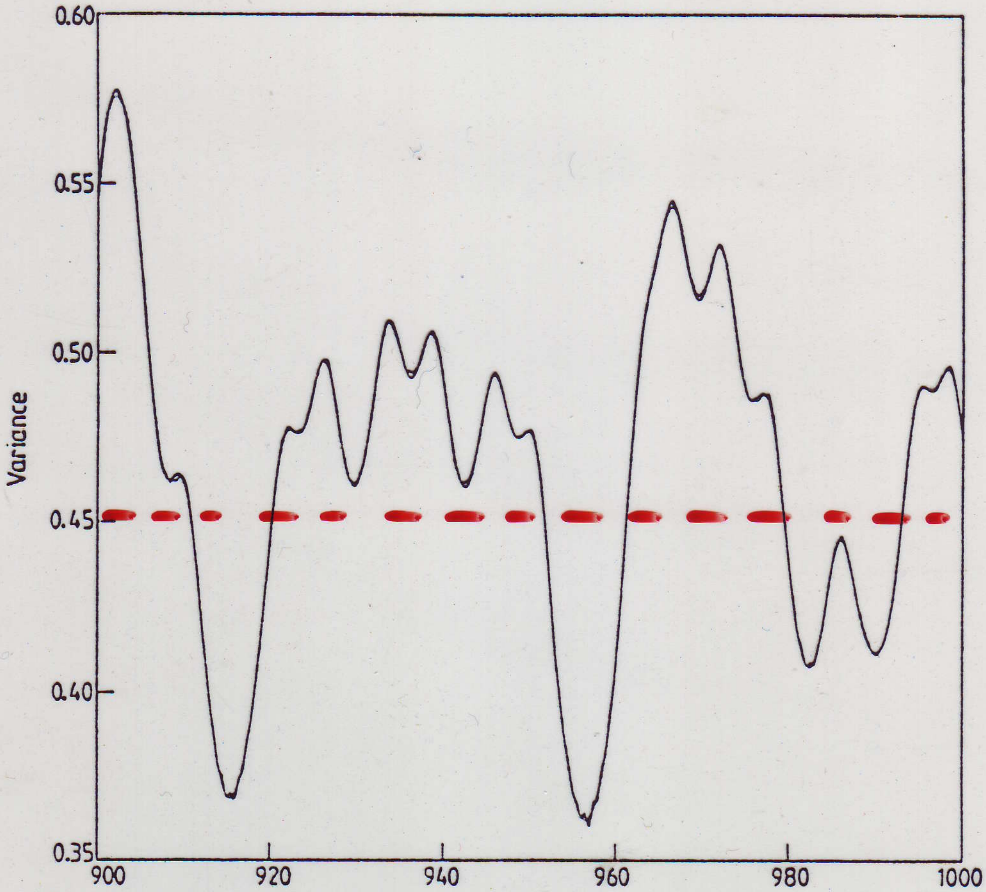
M.V. Berry, Nonlinearity 1 (1988) 399

ζ(s) : number variance . Zeros from Odlyzko

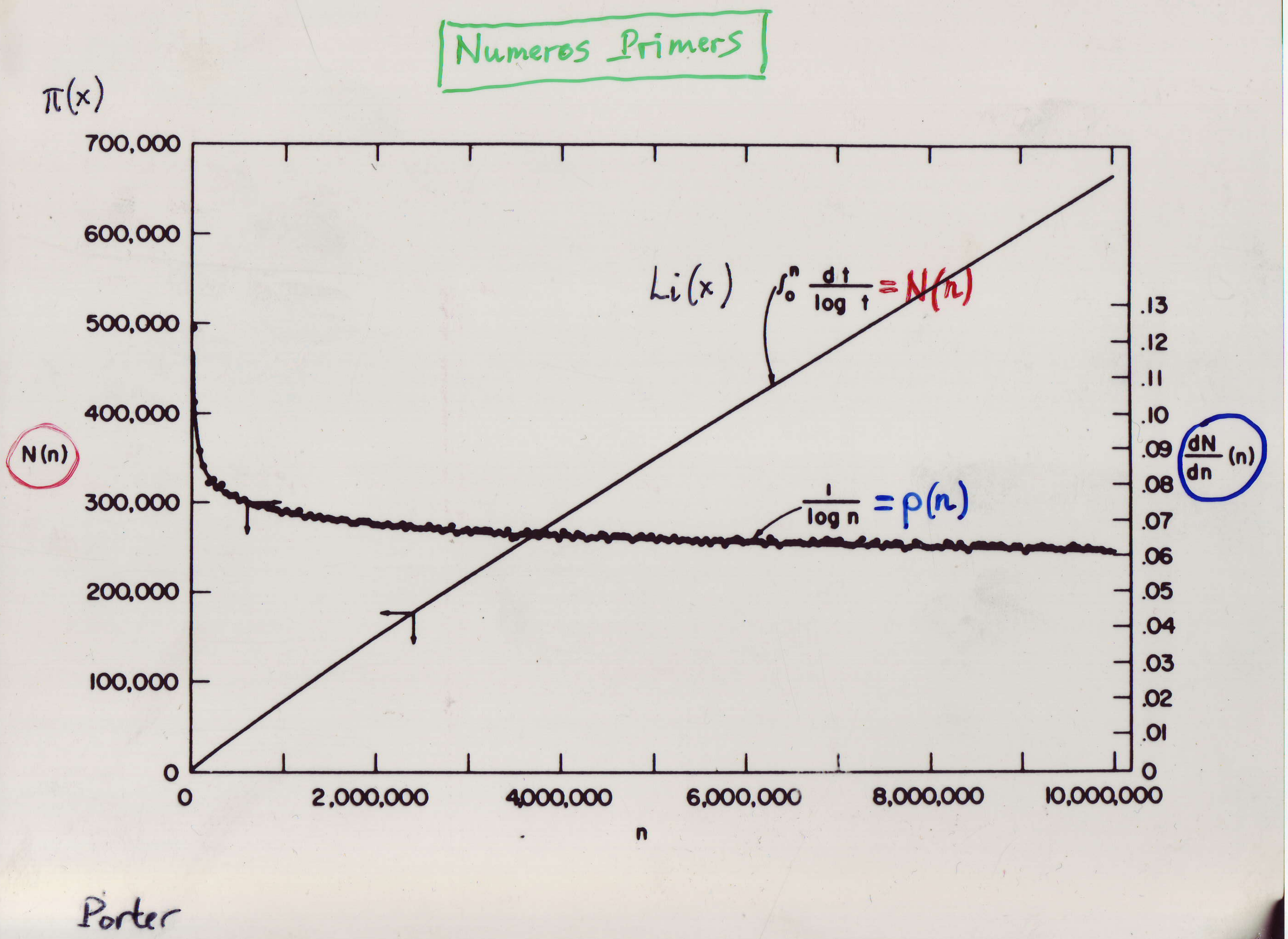
Σ^2



Σ^2



M.V. Berry, Nonlinearity 1 (1988) 399

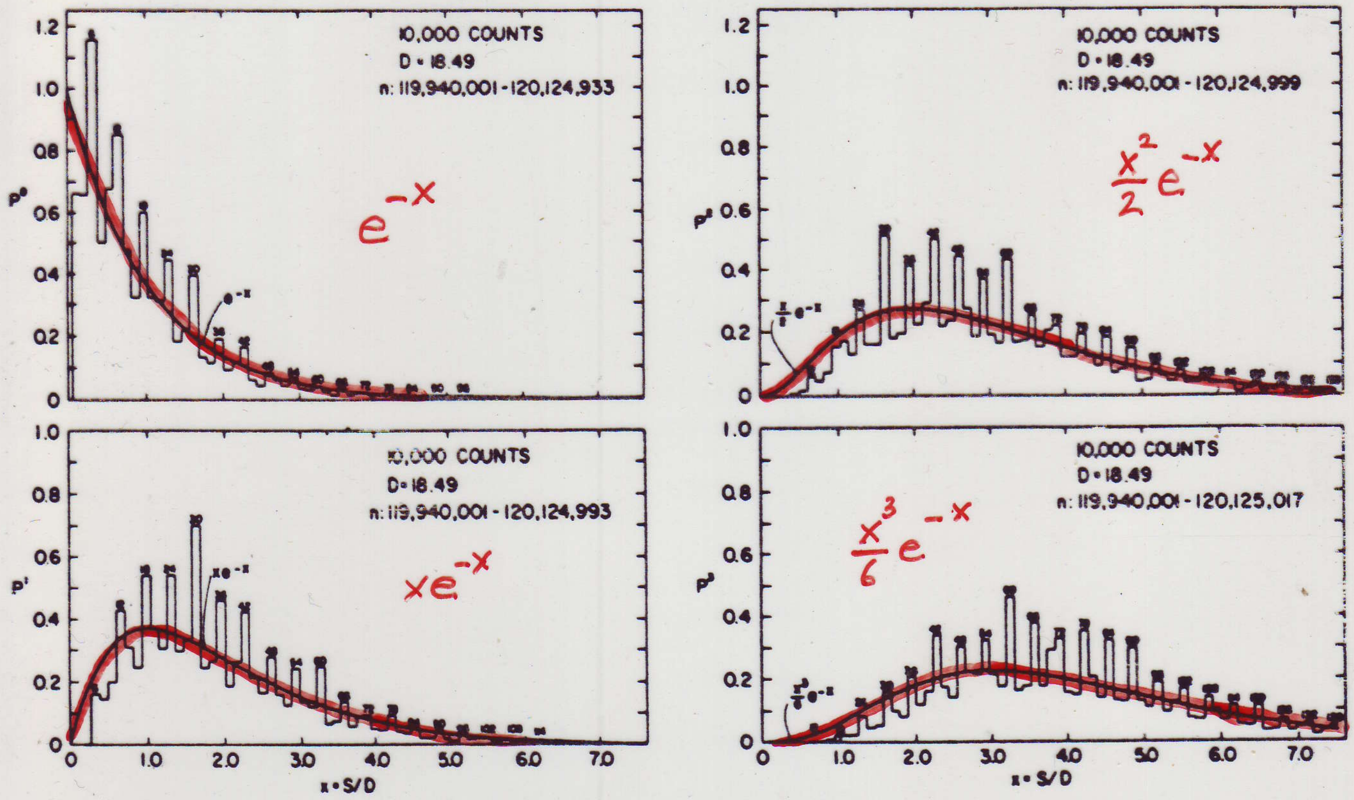


Numeros Primers
π(x)
700,000
600,000
500,000
400,000
300,000
200,000
100,000
0
N(n)
Li(x)
$\int_0^n \frac{dt}{\log t} = N(n)$
$\frac{1}{\log n} = p(n)$
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
$\frac{dN}{dn}(n)$
0
2,000,000
4,000,000
6,000,000
8,000,000
10,000,000
n
Porter

Prime numbers

Spacing distributions

nearest neighbours



Poisson:

$$p(k,x) = \frac{x^k}{k!}e^{-x}$$

C. Porter

Bogomolny & Keating
Nonlinearity $\underline{8}$ (1995) 115
$\underline{9}$ (1996) 911

Pair distribution of prime numbers (prime correlations)

$\pi_2(k;x)$: # of primes $p \le x$ such that $p+k$ is also prime

Hardy-Littlewood conjecture (1923)

For large x

$$\pi_2(k;x) \sim \frac{x}{\log^2 x} \, C(k)$$

$$C(k) = \begin{cases} 2 \underbrace{\prod_{q>2} \left(1 - \frac{1}{(q-1)^2}\right)}_{\approx 0.66016\ldots} \cdot \prod_{\substack{p>2 \\ p|k}} \left(\frac{p-1}{p-2}\right) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

odd primes

odd prime divisors of k

On the average

$$C(k) \sim \left(1 - \frac{1}{2|k|}\right)$$

Would primes be uncorrelated $\quad C(k) = 1$

Using the 'trace' formula:

Hardy-Littlewood conjecture $\Longrightarrow$ Montgomery-Odlyzko conjecture

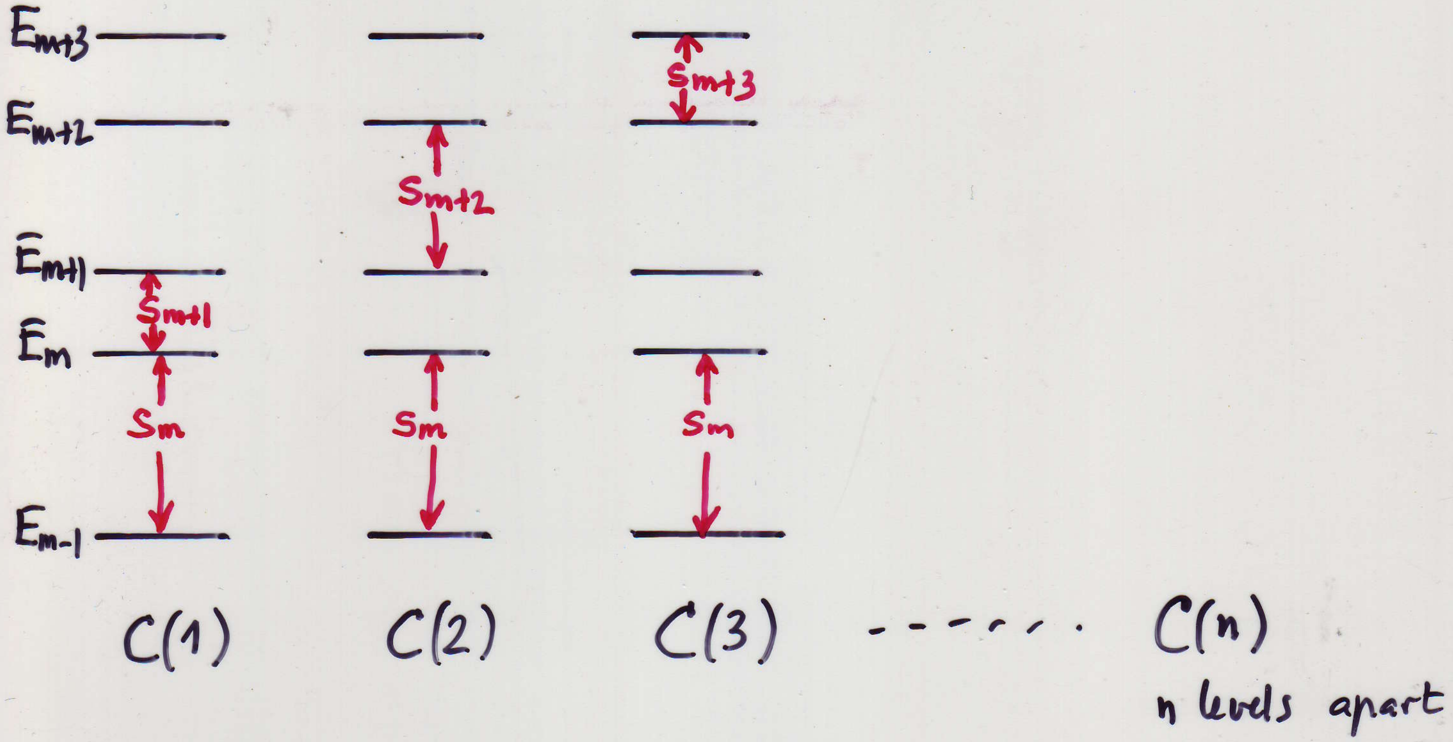
'classical' 'quantum'

Some of the semiclassical formulae are exact for Riemann's ζ

{ Montgomery-Odlyzko (GUE or CUE hypothesis) } and { Hardy-Littlewood conjecture }

are consistent

(periodic orbit theory)



E_{m+3}
E_{m+2}
E_{m+1}
E_m
E_{m-1}
S_{m+1}
S_m
S_{m+2}
S_{m+3}
C(1)
C(2)
C(3)
C(n)
n levels apart

consecutive eigenvalues, denoted $s_m = x_m - x_{m-1}$

the $\boxed{\text{autocovariances } C(n)}$

$$C(n) = \langle (s_m - \langle s_m \rangle)(s_{m+n} - \langle s_{m+n} \rangle) \rangle = \langle s_m s_{m+n} \rangle - 1$$

$$= I(n) - 1$$

where $I(n)$ are the spacing autocorrelations

$$C(n) = \frac{1}{2}\left[\sigma^2(n+1) - 2\sigma^2(n) + \sigma^2(n-1)\right] \qquad n \geq 2$$

$$C(1) = \frac{1}{2}\left[\sigma^2(2) - 2\sigma^2(1)\right]$$

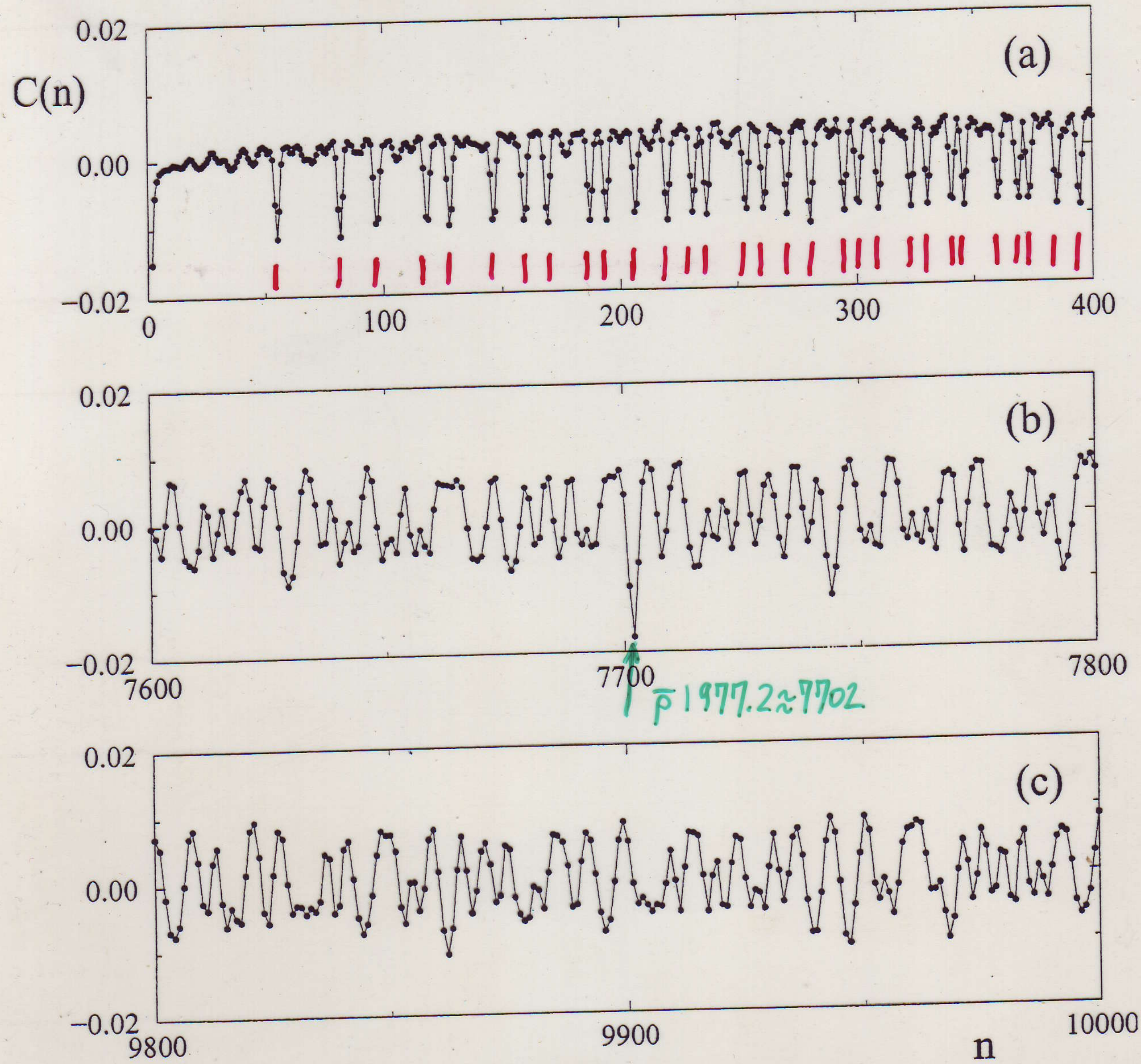
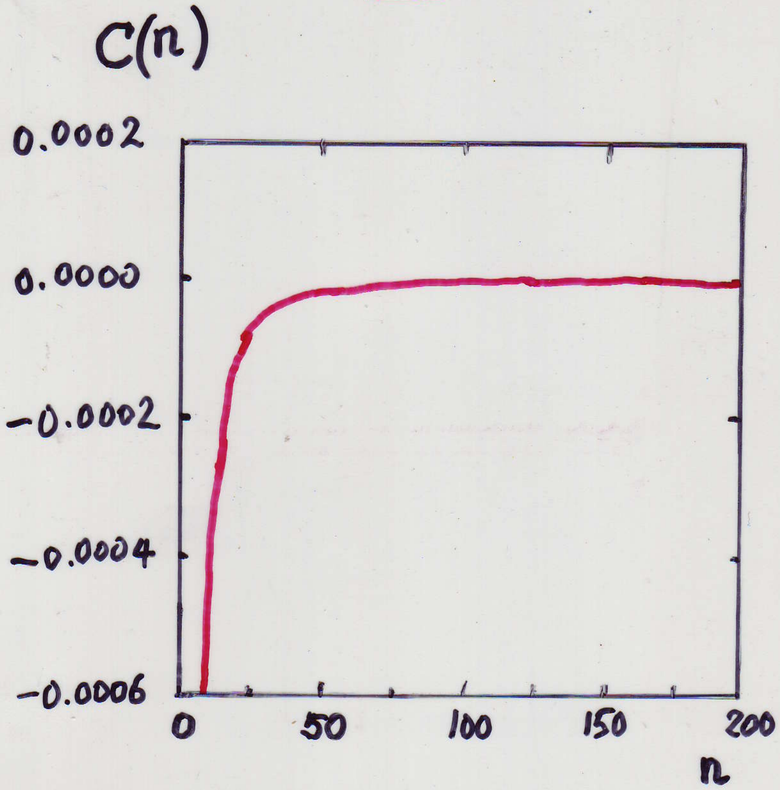


Fig. 4. Spacing autocovariances $C(n)$ for Riemann zeros for three different ranges of n to illustrate several regimes and phenomena. Circles, from theory (Eq. (3.1), see caption of Fig. 2). The last part of Fig. 4c corresponds to Fig. 3.

RMT



$\beta=2$

Leading order

$$C(n) = \frac{1}{\beta\pi^2} \log\left(1 - \frac{1}{n^2}\right)$$

Predictions

- Important differences / RMT from $n \approx 3$ on

- For small values of n, typical values of fluctuations 10^{-3}

- For larger values of n, amplitude of oscillations typically $\sim 10^{-2}$

- For $n > \frac{1}{\tau_{min}} \approx 35$, zeros of Riemann are expected to produce isolated peaks, whose height is almost constant, of order
$$\frac{\pi^2}{2}\bar{\rho}^2(x) \approx 0.013$$

- Isolated peaks should disappear around
$$\bar{\rho}(n) \approx 1 \quad (n \approx 3300)$$

Riemann's zeros

$$C(n) = \frac{2}{\pi^2} \sum_{\substack{p,\,r \\ \text{primes}}} \frac{\sin^2(\pi r \tau_p)}{r^2 p^r} \cos(2\pi n r \tau_p)$$

$$Z(s) = \prod_p (1 - p^{s-1}) = \zeta^{-1}(1-s)$$

ζ : 'quantum' <u>and</u> 'classical'

$$F(s) = \prod_{r=2} [\zeta(r - rs)]^{-(r-1)/r^2}$$

$$\boxed{C(n) = -\frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k)!} \operatorname{Re} \frac{\partial^{2k}}{\partial n^{2k}} \log \frac{\zeta(1 - in/\bar{\rho})}{\prod_{r=2}^{\infty} [\zeta(r - inr/\bar{\rho})]^{(r-1)/r^2}}}$$

$$C(n) = \frac{1}{2\pi^2} \sum_{r=1} \left\{ \frac{(\delta_{r,1} + 1 - r)}{r^2} \left[\log \left| 1 - \frac{1}{(n + in_{0r})^2} \right| \right. \right.$$

$$\left. \left. - \sum_{\mu} \log \left| 1 - \frac{1}{(n + in_{\mu r})^2} \right| - \sum_{m} \log \left| 1 - \frac{1}{(n + in_{mr})^2} \right| \right] \right\}$$

contribution:

$n_{0r} = \bar{\rho}(1 - 1/r)$ ← pole

$n_{\mu r} = \bar{\rho}[1 - (1/2 - it_{\mu})/r]$ ← critical zeros

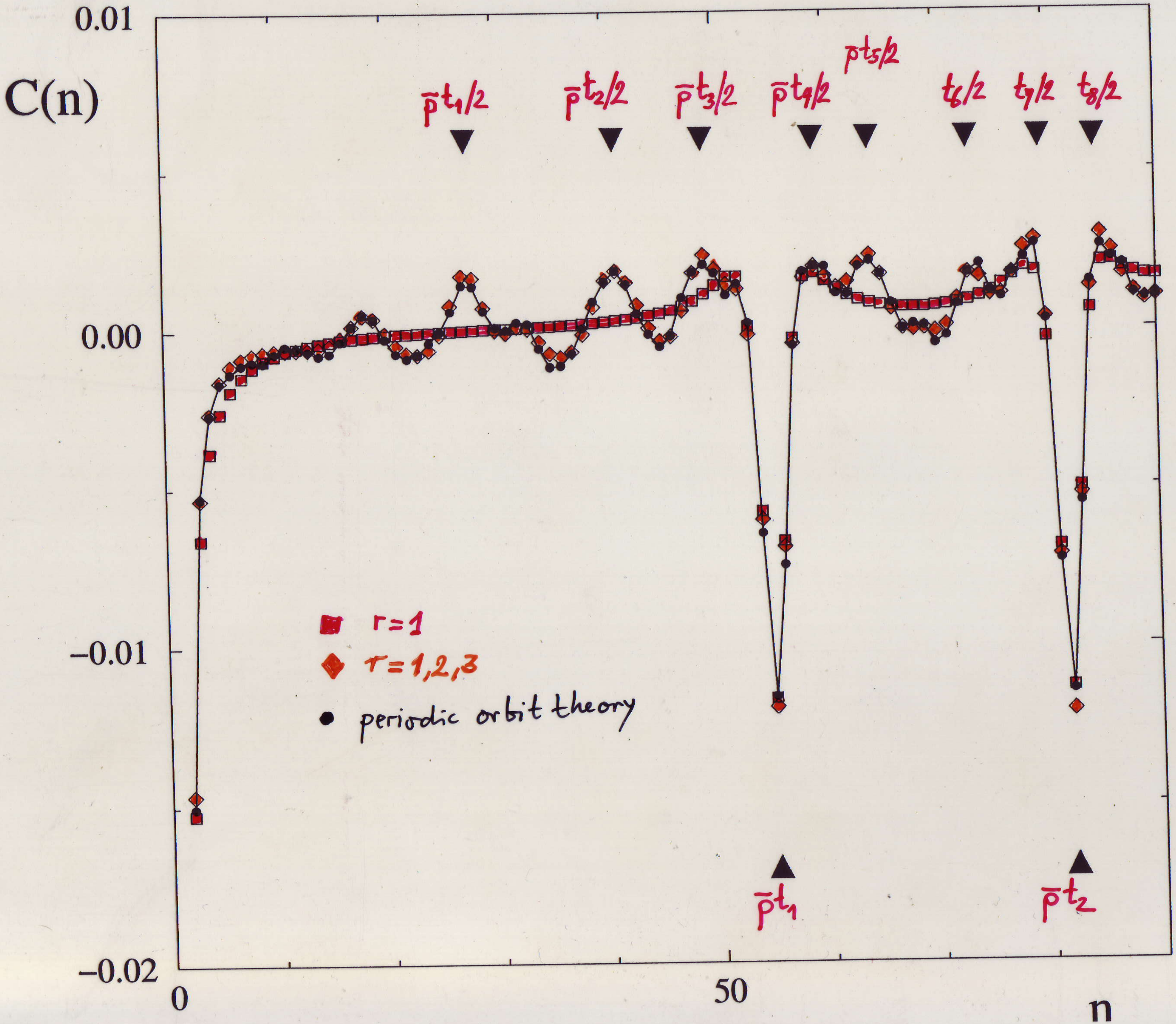
$n_{mr} = \bar{\rho}[1 + 2(m + 1)/r]$ ← trivial zeros of ζ

$m = 1, 2, \ldots$

The $r=1$ term of the pole at 1 of ζ produces de RMT leading result

$$C(n) = \frac{1}{\beta\pi^2} \log\left(1 - \frac{1}{n^2}\right)$$

Bohigas, Leboeuf, Sanchez Found. Phys. 31 (2001) 489



$\zeta(\frac{1}{2}+it_\mu)=0$

around 10^{12}-th zero

$X = 267653395648.8475$

$\bar{p} = \frac{1}{2\pi}\log\left(\frac{X}{2\pi}\right) \approx 3.895\ldots$

Leboeuf
Sanchez
OB

Found. Phys. 31 (2001) 489

Riemann's zeros

$\tilde{N}$ is gaussian distributed (Selberg)

$$\langle \tilde{N}^2 \rangle = \frac{1}{2\pi^2}\left[\log\log\left(\frac{E}{2\pi}\right) + 1.4009\right]$$

This can be recovered by periodic orbit theory

RMT gives for $\tilde{\Omega}$ a limiting Gaussian distribution, which is incorrect

Periodic orbit theory, however, gives the correct answer.

"Riemannium"

Leboeuf, Monastra, OB
Reg. Chaot. Dyn. 6 (2001) 205
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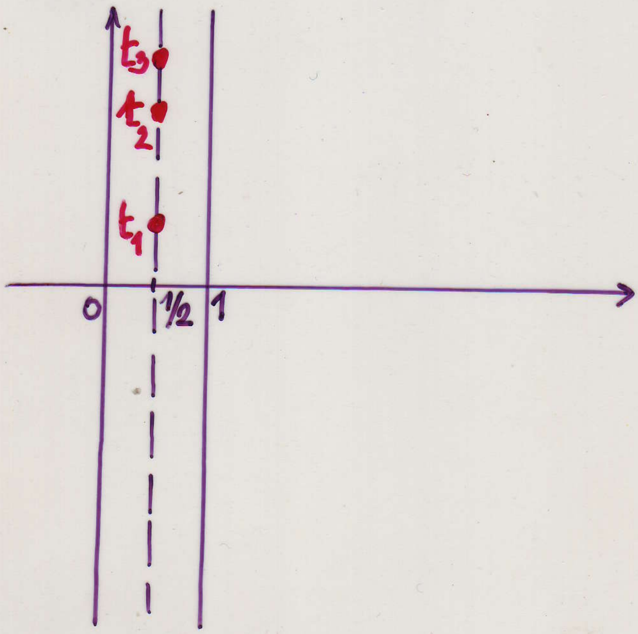
Consider the zeros of $\zeta(s)$ on the critical line

$$\zeta\left(\tfrac{1}{2}+it_n\right)=0 \qquad n=1,2,\ldots$$



Take the ordered sequence $\{t_i\}$ $\quad t_i>0$

and form

$$\Omega=\sum_{i\in E_F} t_i$$

Ω can be interpreted as the total energy of a system of (spinless) fermions (non-interacting) whose single-particle energies are the t_i's.

Tails
$\sim e^{-c x^2 \log^2 |x|}$
(Bleher)

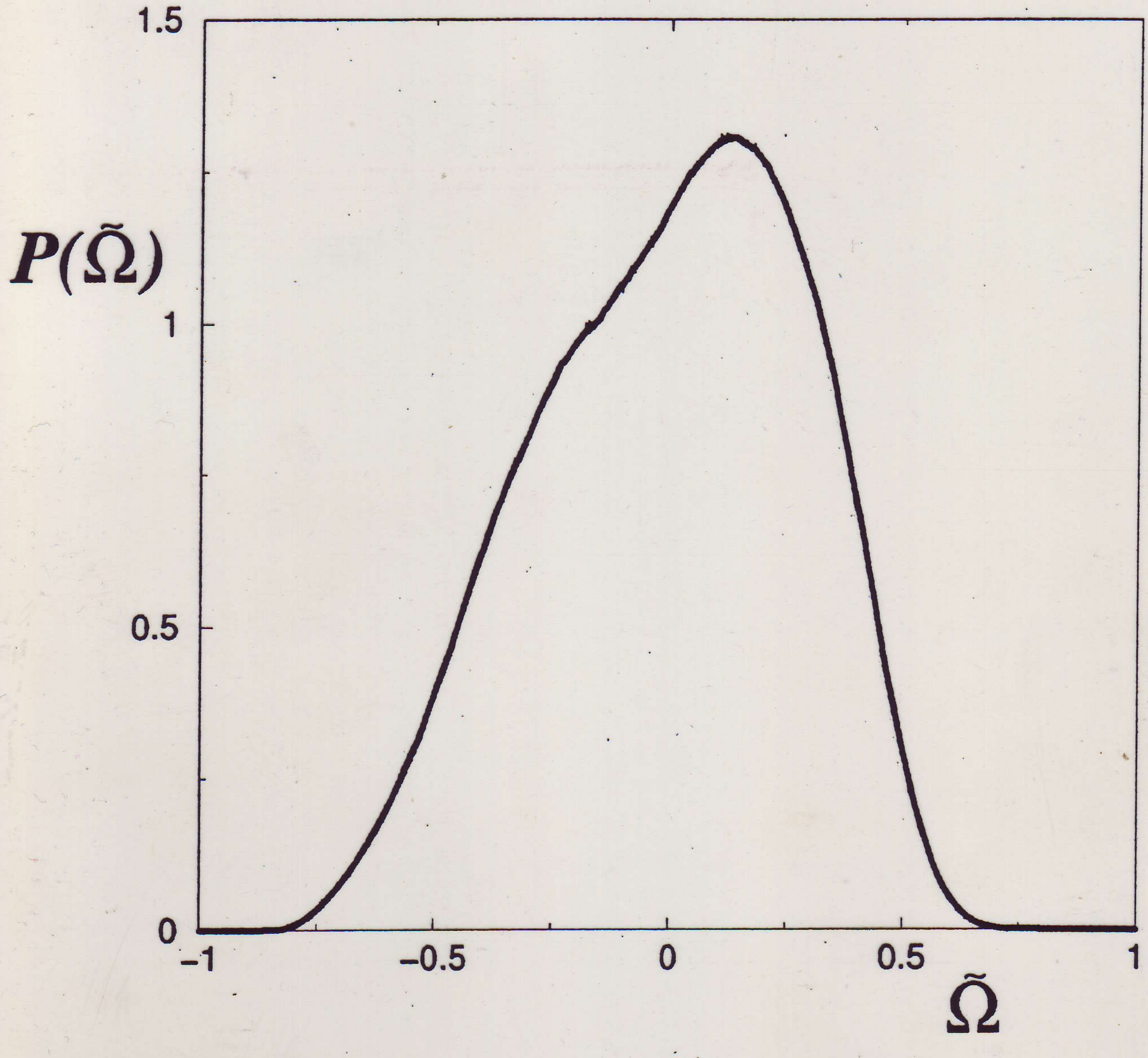


Fig. 1. Distribution of $\widetilde{\Omega}$ computed numerically at $E_F \approx 1.44 \times 10^{20}$ (results based on data from A. Odlyzko)

Leboeuf, Monastra, O. B.
Reg. Chaot. Dyn. 6 (2001) 205
nlin. CD/0101014

Autocorrelation of $\tilde{\Omega}$

$$C_{\Omega}(\epsilon) = \frac{1}{\langle \tilde{\Omega}_0^2 \rangle} \langle \tilde{\Omega}(E_F)\tilde{\Omega}(E_F+\epsilon) \rangle$$

$$\langle \widetilde{\Omega}_0^2 \rangle = \frac{1}{2\pi^2} \sum_p \sum_{r=1}^{\infty} \frac{1}{r^4 p^r \log^2 p} \approx 7.9 \times 10^{-2}$$

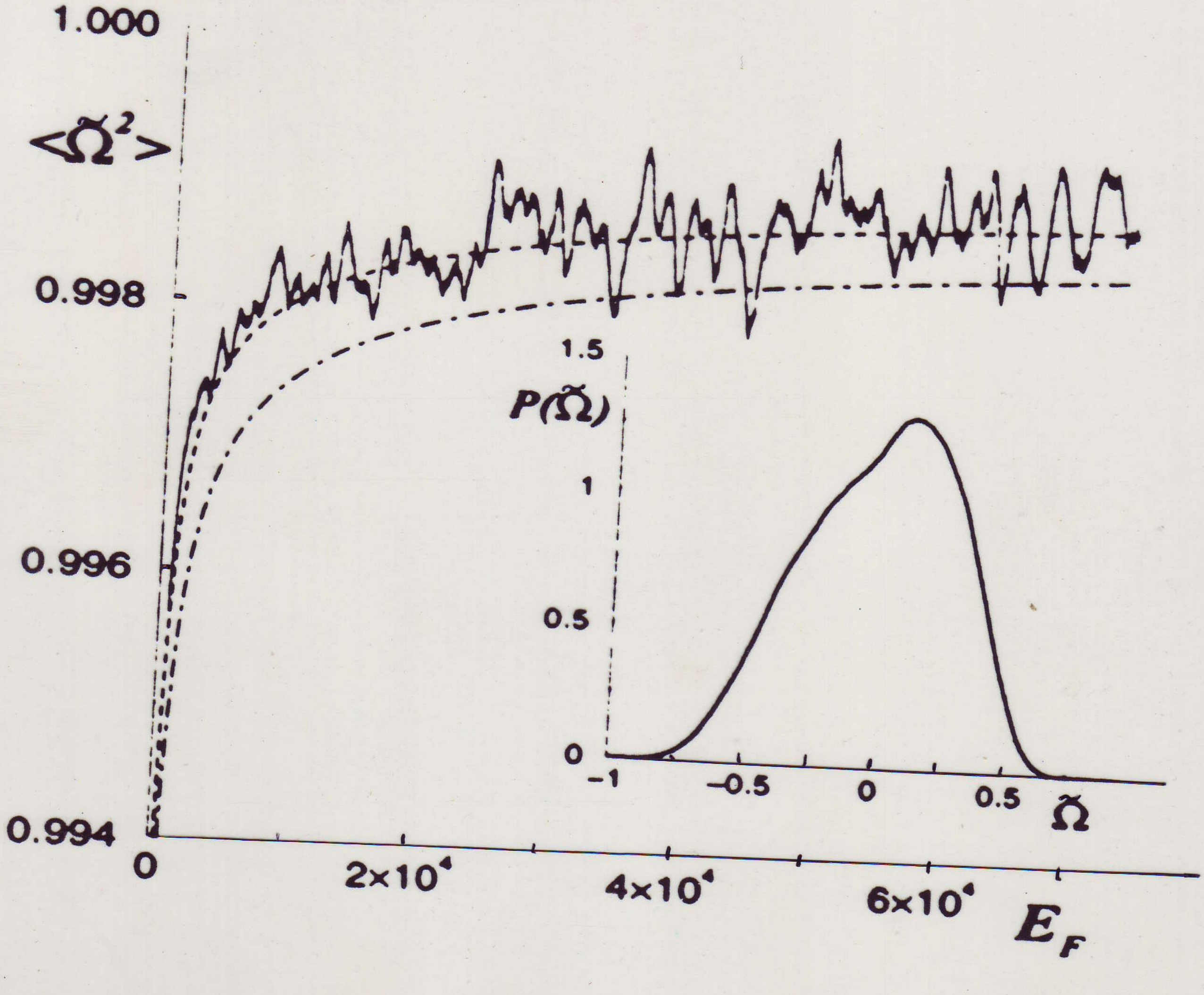
$$\langle \widetilde{\Omega}^2 \rangle = \langle \widetilde{\Omega}_0^2 \rangle - \frac{1}{12\pi^2 \log^2(E_F/2\pi)}$$

$$\langle \widetilde{\Omega}_0^3 \rangle =$$

$$-\frac{3}{4\pi^3} \sum_p \sum_{r_i, r_j = 1}^{\infty} [r_i^2 r_j^2 (r_i + r_j)^2 p^{r_i + r_j} \log^3 p]^{-1} \approx$$

$$\approx -5.78 \times 10^{-3}$$

$$C_\Omega(\varepsilon) = \frac{1}{2\pi^2 \langle \widetilde{\Omega}_0^2 \rangle} \sum_p \sum_{r=1}^{\infty} \frac{\cos(\varepsilon r \log p)}{r^4 p^r \log^2 p}$$



Leboeuf
Monastra
OB

Moment	Semiclassics	Numerics
2	7.9290×10^{-2}	7.928×10^{-2}
3	-5.7822×10^{-3}	-5.785×10^{-3}
4	1.4814×10^{-2}	1.481×10^{-2}
5	-2.7787×10^{-3}	-2.776×10^{-3}
6	4.0007×10^{-3}	4.001×10^{-3}

$$\widetilde{\Omega}_{CUE} = (1/\pi)\mathrm{Re}\sum_k (\mathrm{Tr}U^k/k^2)exp(-ik\theta)$$

$$\langle\widetilde{\Omega}^2_{CUE}\rangle = (1/2\pi^2)\sum_k \langle|\mathrm{Tr}U^k|^2\rangle/k^4$$

$$\langle\widetilde{\Omega}^2_{CUE}\rangle = \frac{\zeta(3)}{2\pi^2} - \frac{1}{12\pi^2N^2} + \mathcal{O}(1/N^4)$$

6.1×10^{-2}

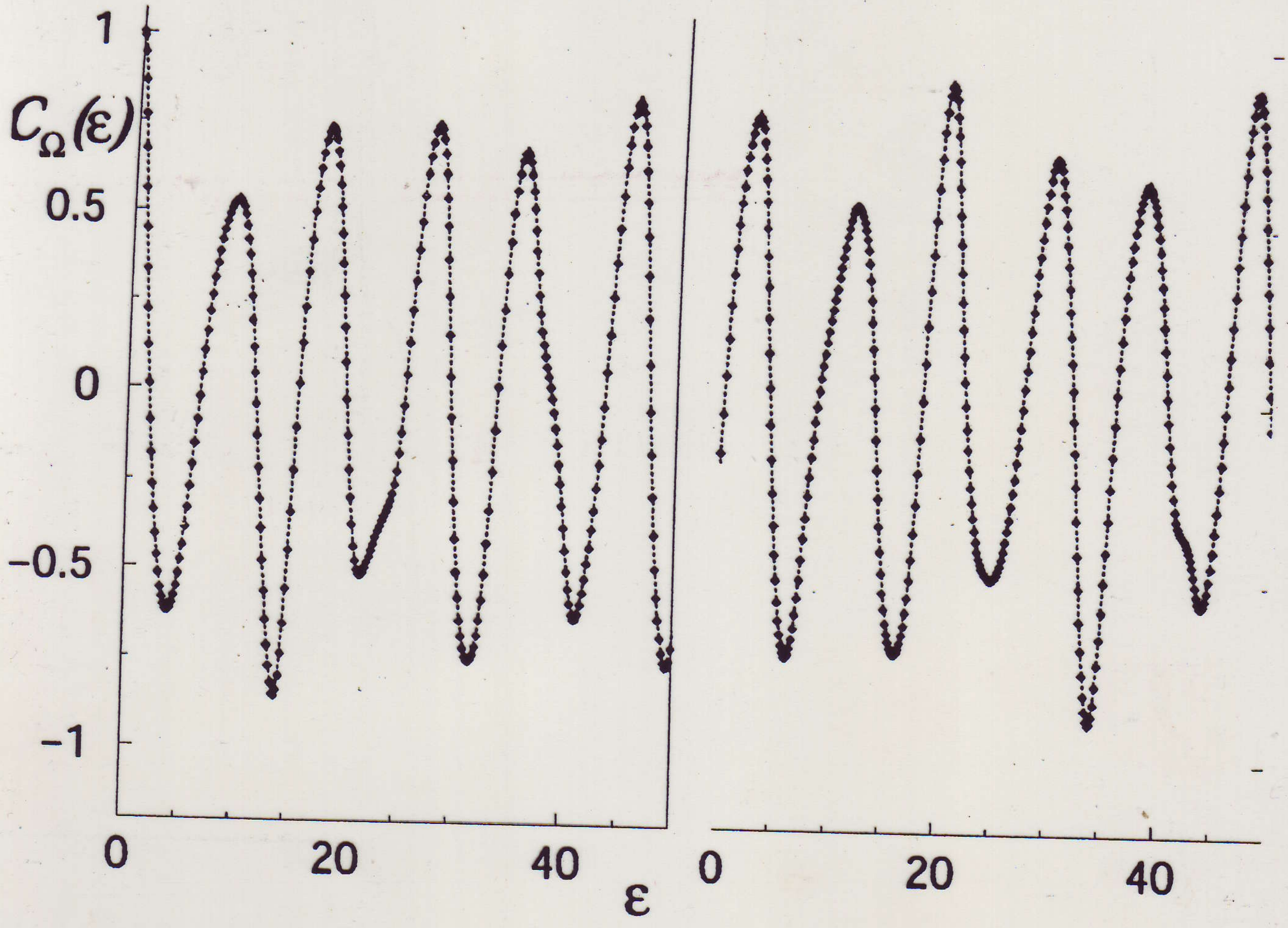
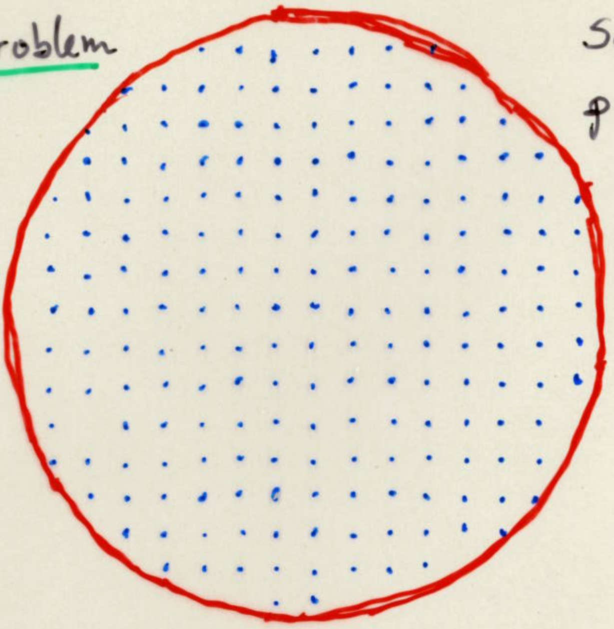


Fig. 2. The autocorrelation Eq.(8) (dotted line) compared to numerical data (dots). E_F as in Fig. 1. On the right part a constant $\varepsilon_0 \approx 1.29 \times 10^{20}$ should be added to the abscissa

Spectrum : $E_{m,n} = m^2 + n^2$

Circle problem

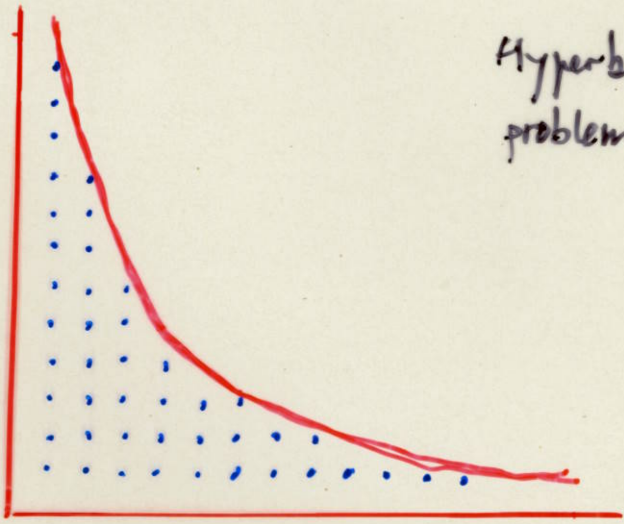
Square billiard problem



Spectrum $E_{m,n} = m \cdot n$

Dirichlet divisor problem

Hyperbola billiard problem



Keating & Snaith, CMP 214 (2000) 57

Long standing conjecture:

moments

$$\lim_{T\to\infty} \frac{1}{(\log T)^{\lambda^2}} \frac{1}{T} \boxed{\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2\lambda} dt} = f(\lambda)\, a(\lambda)$$

where $a(\lambda) = \prod_p \left\{ \left(1-\frac{1}{p}\right)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)} \right)^2 p^{-m} \right) \right\}$

Assumption : $f(\lambda)$ can be modelled by RMT (CUE)

$N \times N$ unitary matrices

$$P(\vartheta_1, \ldots, \vartheta_N) = \frac{1}{(2\pi)^N N!} \prod_{j<m} \left| e^{i\vartheta_j} - e^{i\vartheta_m} \right|^2$$

Characteristic polynomial of unitary matrices

$$Z(\vartheta) = \det\left(I - U e^{-i\vartheta}\right) = \prod_{n=1}^{N} \left(1 - e^{i(\vartheta_n - \vartheta)}\right)$$

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \left\langle |Z(\vartheta)|^{2\lambda} \right\rangle_{U(N)} \equiv f_{CUE}(\lambda)$$

$f(\lambda)$

λ	RMT	Riemann ζ
1	1	1 (th^m: Hardy-Littlewood 1918)
2	$\frac{1}{12}$	$\frac{1}{12}$ (th^m: Ingham 1926)
3	$\frac{42}{9!}$	$\frac{42}{9!}$ (conjecture: Conrey & Gosh 1992)
4	$\frac{24024}{16!}$	$\frac{24024}{16!}$ (conjecture: Conrey & Gonek 1998)
λ	$\boxed{\prod_{n=0}^{\lambda-1} \frac{n!}{(\lambda+n)!}}$	

$\therefore$ RMT is able to guess (presumably) exact results previously unknown from mathematicians concerning the Riemann ζ.

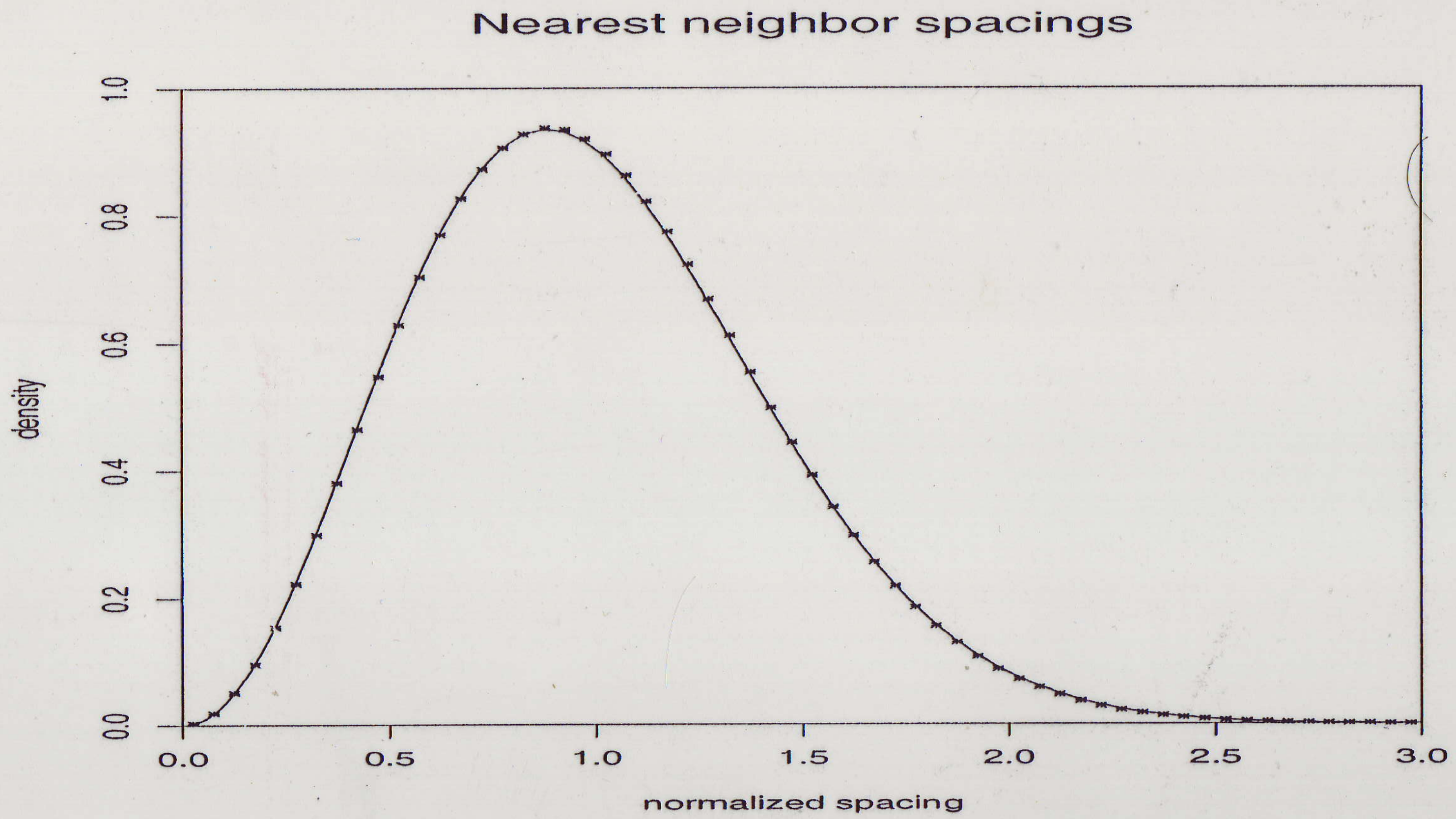
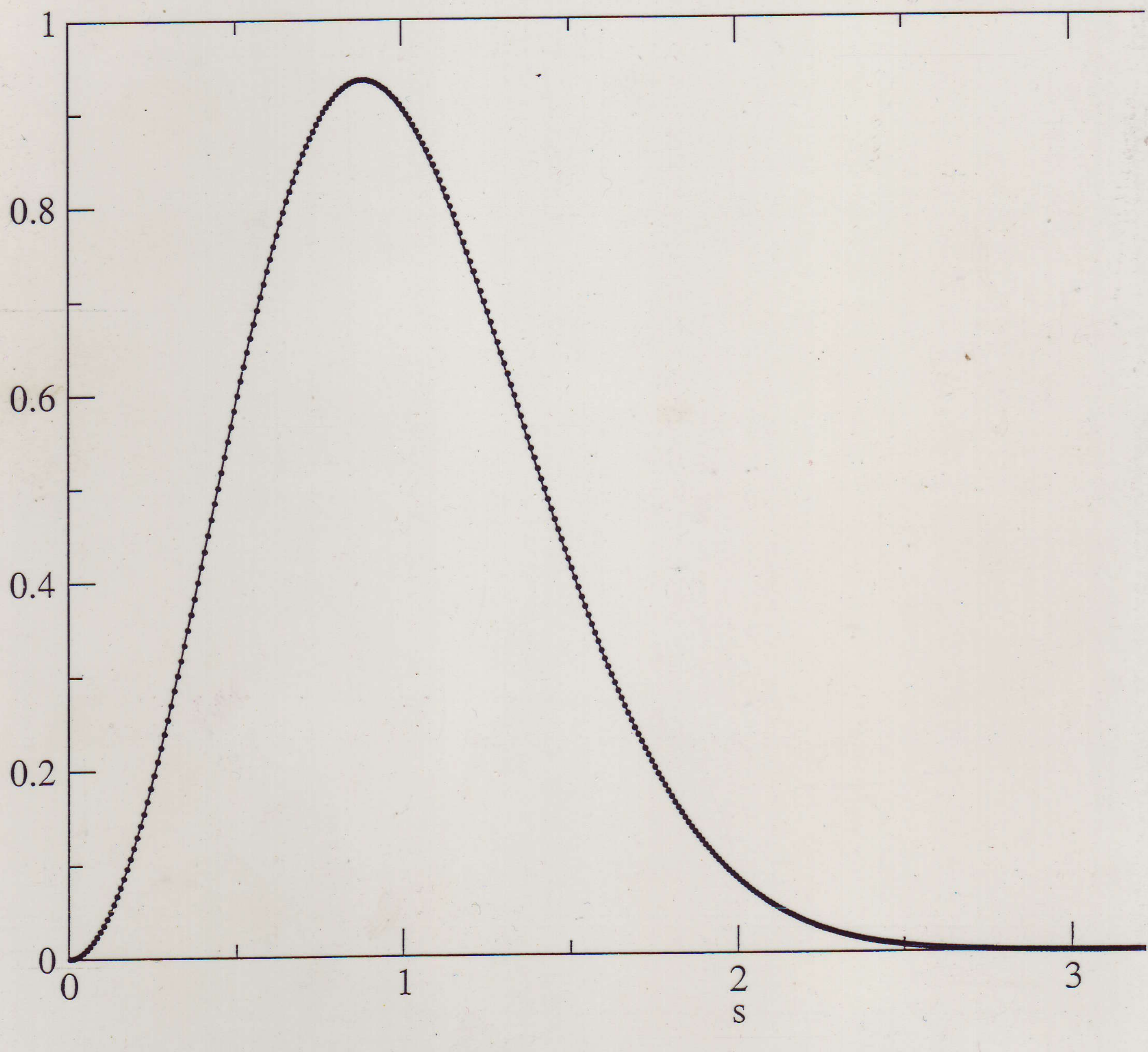


Figure 1. Probability density of the normalized spacings δ_n. Solid line: $\boxed{\text{Gue prediction.}}$ Scatterplot: empirical data based on a billion zeros near zero # $1.3 \cdot 10^{16}$.

$\beta=2$

A. Odlyzko

10^{23}



asymptotic $\beta=2$

$p_S(s) - p^0(s)$

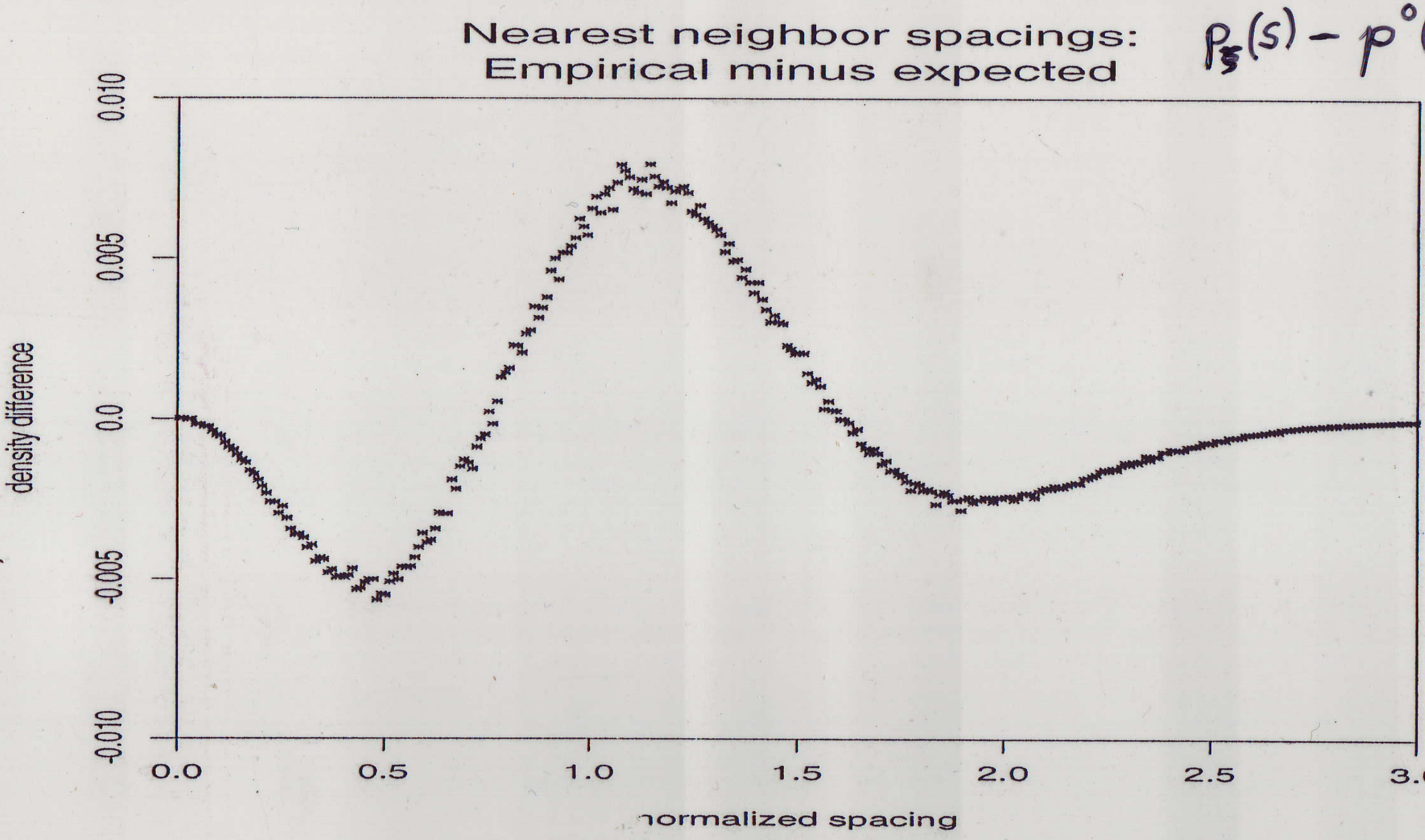


Figure 2. Probability density of the normalized spacings δ_n. Diffrence between empirical distribution for a billion zeros near zero # $1.3 \cdot 10^{16}$ and the GUE prediction.

A. Odlyzko

CUE:

Spacing distribution for $N \times N$ $\quad p^{(N)}(s)$

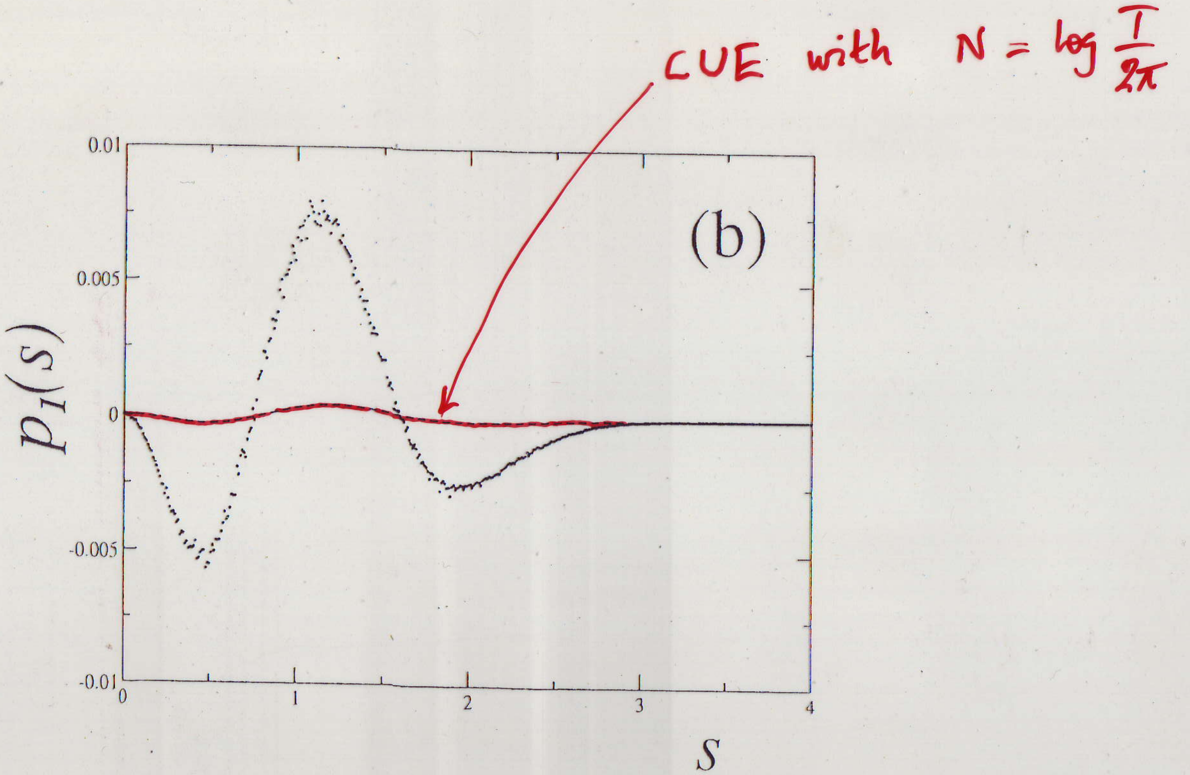
$$\boxed{p^{(N)}(s) = p_0(s) + \frac{1}{N^2} p_1(s) + \frac{1}{N^4} p_2(s) + \dots}$$

$$p^{(N)}(s) = \frac{d^2 E^{(N)}(s)}{ds^2}$$

$$E^{(N)}(s) = \det\left[\delta_{jk} - \frac{\sin\left(\frac{\pi s}{N}(j-k)\right)}{\pi(j-k)}\right]_{1 \leq j,k \leq N}$$

GUE

$$p^{(N)}(s) = p_0(s) + \frac{(-1)^{N+1}}{N} p_1(s) + O(1/N^2)$$


CUE with $N = \log \frac{T}{2\pi}$
(b)
$p_1(s)$
0.01
0.005
0
-0.005
-0.01
0
1
2
3
4
s

Bogomolny & Keating

$$r_2(\epsilon) = \bar{\rho}^2 + r_2^{(diag)}(\epsilon) + r_2^{(off)}(\epsilon)$$

$$\bar{\rho} = \frac{1}{2\pi} \ln\left(\frac{E}{2\pi}\right)$$

$$r_2^{(diag)}(\epsilon) = -\frac{1}{4\pi^2}\frac{\partial^2}{\partial\epsilon^2}\left[\ln|\zeta(1+i\epsilon)|^2\right] -$$

$$-\frac{1}{4\pi^2}\sum_p\left(\frac{\ln^2 p}{(p^{1+i\epsilon}-1)^2} + c.c.\right)$$

$$r_2^{(off)}(\epsilon) = \frac{1}{4\pi^2}|\zeta(1+i\epsilon)|^2 e^{2\pi i\bar{\rho}\epsilon}\prod_p\left[1 - \frac{(p^{i\epsilon}-1)^2}{(p-1)^2}\right] + c.c.$$

In units of local mean spacing

$$R_2(s) = \frac{1}{\bar{\rho}^2}\, r_2\left(\frac{s}{\bar{\rho}}\right)$$

ζ - function

Bogomolny, Bohigas, Leboeuf, Monastra
J. Phys. A39 (2006) 10743
math.NT/0602270

$$R_2(s) = 1 - \frac{\sin^2(\pi s)}{(\pi s)^2} - \frac{\beta}{\bar{\rho}^2 \pi^2} \sin^2(\pi s)$$

$$- \frac{\delta}{\bar{\rho}^3 2\pi^2} s \cdot \sin(2\pi s) + O(\bar{\rho}^{-4})$$

$$= 1 - \frac{\sin^2(\pi s)}{(\pi s)^2} - \frac{\beta}{\bar{\rho}^2 \pi^2} \sin^2(\pi \alpha s) + O(\bar{\rho}^{-4})$$

$$p(s) = p_\infty^{CUE}(s) + \frac{1}{N_{eff}^2} \, p^1(\alpha s)$$

$$\bar{\rho} = \frac{1}{2\pi} \log \frac{E}{2\pi}$$

$$\alpha = 1 + \frac{\delta}{\beta} \frac{1}{2\pi\bar{\rho}}$$

$$\beta = \gamma_0^2 + 2\gamma_1 + \sum_p \frac{\log^2 p}{(p-1)^2} = 1.57314\ldots$$

$$\delta = \sum_p \frac{\log^3 p}{(p-1)^2} = 2.3357$$

$$N_{eff} = \frac{1}{\sqrt{12\beta}} \log \frac{E}{2\pi} = 0.23\ldots \log \frac{E}{2\pi}$$

Stieltjes constants γ_m

$$\gamma_m = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{\ln^m r}{r} - \frac{\ln^{m+1} n}{m+1}\right)$$

$$\gamma_0 = \gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \frac{\ln n}{1}\right)$$

$$\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\gamma_r(z-1)^r$$

$$\gamma_0 = \gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \ln n\right)$$

$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{\ln r}{r} - \frac{\ln^2 n}{2}\right)$$

Bogomolny, Bohigas
Leboeuf, Monastra

math.NT/0602270

J. Phys. A39 (2006) 10743

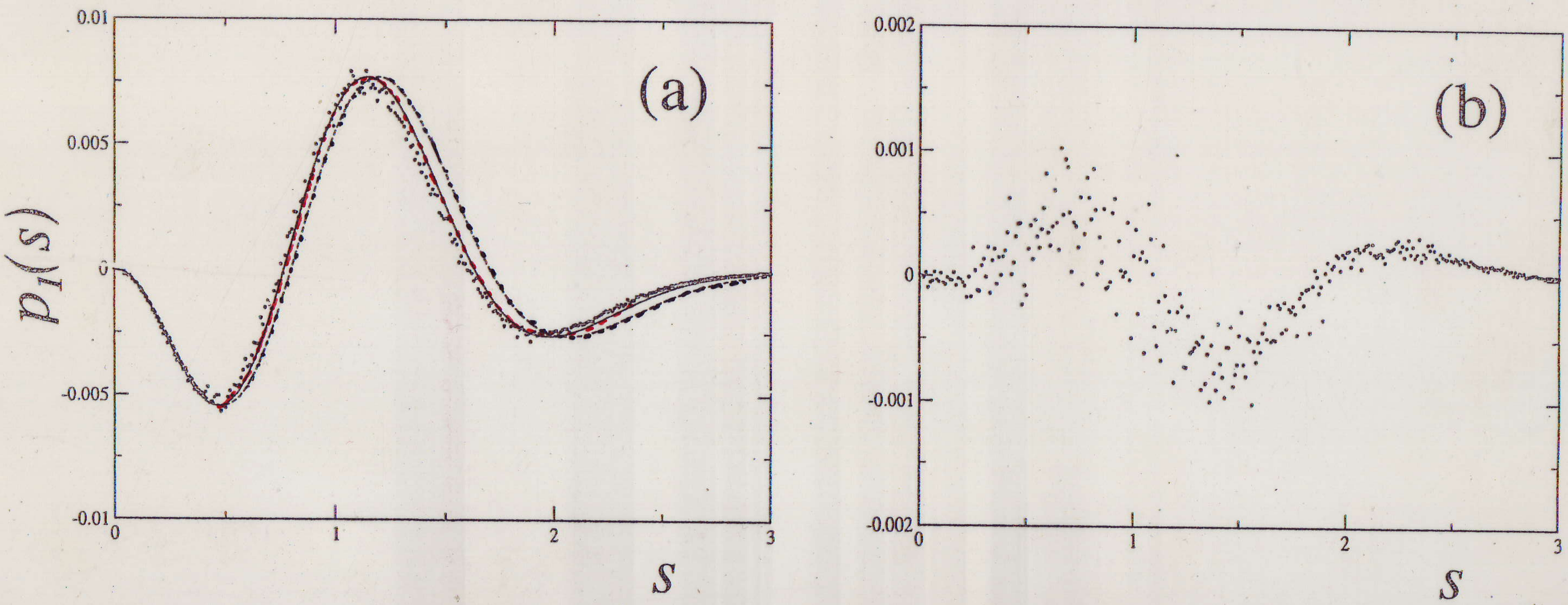


Figure 2: (a) Difference between the nearest neighbor spacing distribution of the Riemann zeros and the asymptotic CUE distribution for a billion zeros located in a window near $E = 2.5041178 \times 10^{15}$ (dots), compared to the theoretical prediction Eq.(21) (full line). The dashed line does not include the scaling of s. (b) Difference between the numerical Riemann values (dots) and the full curve (theory) of part (a).

$\alpha = 1 + 0.0438$

Lebœuf, Monastra math.NT/0602270

J. Phys. A 39 (2006) 10743

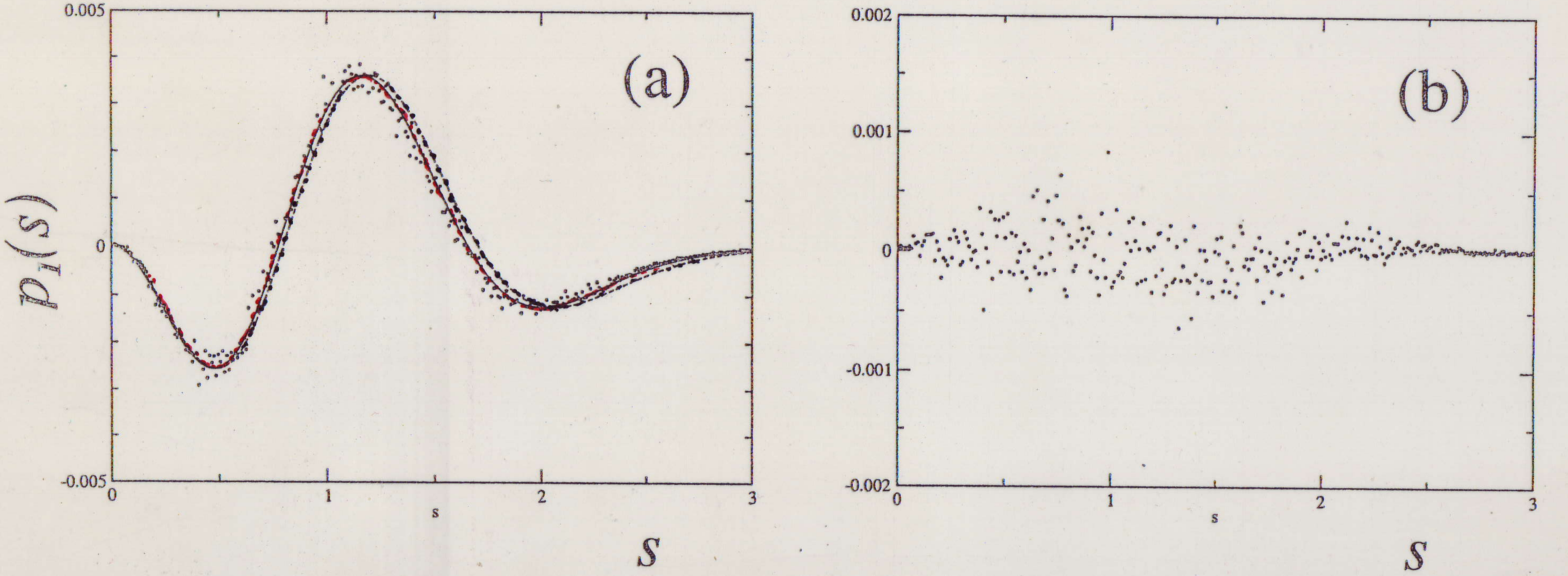


Figure 3: Same as in Fig.2 but for a billion zeros located in a window near $E = 1.30664344 \times 10^{22}$.

$\alpha = 1 + 0.0300$

and consecutive corrections are in even powers of N. Finally, for a CUE ensemble of random matrices, $p(s)$ is expressed as

$$p^{(\mathrm{CUE})}(s) = p_0(s) + \frac{1}{N^2}\, p_1^{(\mathrm{CUE})}(s) + \mathcal{O}(N^{-4}) \;, \tag{20}$$

correlation function of unitary ensembles

$$R_n(x_1, \ldots, x_n) = \det\left(K(x_i, x_j)\right)\Big|_{i,j=1,\ldots,n}$$

GUE in the limit $\qquad K_0(s) = \dfrac{\sin(\pi s)}{\pi s}$

$CUE_N \qquad K_N(x,y) = \dfrac{\sin\left(\pi(x-y)\right)}{N \sin\left(\pi(x-y)/N\right)}$

$$R_2(s) = \begin{vmatrix} 1 & K(s) \\ K(s) & 1 \end{vmatrix}$$

$$R_3(s_1, s_2, s_3) = \begin{vmatrix} 1 & K_{12} & K_{13} \\ K_{21} & 1 & K_{23} \\ K_{31} & K_{32} & 1 \end{vmatrix}$$

$$K_{ij} = K_0(s_{ij}) + k_1(s_{ij}) \qquad s_{ij} = s_i - s_j$$

$$k_1(s) = s \frac{\beta}{2\pi \bar{\rho}^2} \sin(\pi s) + s^2 \frac{\delta}{2\pi \bar{\rho}^3} \cos(\pi s) =$$

$$= \frac{\pi s}{6 N_{eff}^2} \sin(\pi \varkappa s)$$

Hilbert-Polya conjecture

RH is true because imaginary part of zeros of ζ correspond to eigenvalues of a Hermitean operator

relations of ζ with RMT
dynamical interpretation of ζ

reinforce the indications that the Hilbert-Polya conjecture is a promising approach to prove RH

Madrid Intelligencer, Madrid 2006, Springer

Andrew Granville

'It is perhaps fair to say that little has been proved unconditionally about $\zeta(s)$.... but we do have a much better understanding now of many questions to do with zeros and the size of zeta functions... We have seen exciting new methods (from additive combinatorics), prophetic new perspectives (from random matrices)...'