Random Matrix Theory, homework 2, due November 19.

Problem 1. The Circular Unitary Ensemble is a log-correlated random field. Let $(e^{\mathrm{i}\theta_k})_{1\leq k\leq N}$ be the eigenvalues of a Haar-distributed matrix in $\mathrm{U}(N)$. The eigenangles have joint probability distribution

$$\mathbb{P}(\mathrm{d}\boldsymbol{\theta})=\frac{1}{N!}\prod_{1\leq i<j\leq N}|e^{\mathrm{i}\theta_i}-e^{\mathrm{i}\theta_j}|^2\frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_N}{2\pi}.$$

(i) Prove that $\chi=\sum_{k=1}^N\delta_{\theta_k}$ is a determianntal point process with correlation kernel

$$K(x,y)=K^{(N)}(x,y)=\frac{1}{2\pi}\frac{\sin N\frac{x-y}{2}}{\sin\frac{x-y}{2}}$$

with respect to the Lebesgue measure on $(0,2\pi)$.

(ii) Let $\phi:[0,2\pi)\to\mathbb{R}$ be bounded measurable. Prove that

$$\mathbb{E}\prod_{k=1}^N(1+\phi(\theta_k))=\sum_{n\geq 0}\frac{1}{n!}\int_{(0,2\pi)^n}\prod_{j=1}^n\phi(x_j)\det_{n\times n}K(x_i,x_j)\mathrm{d}x_1\dots\mathrm{d}x_n.$$

You will need to explain why the right hand side converges.

(iii) Read Section 3 in the book *Trace ideals and applications*.

(iv) Let $A\subset[0,2\pi)$ be measurable. On $\mathrm{L}^2(A)$, define $K\phi$ the convolution operator with kernel $K\phi$, where ϕ is bounded measurable:

$$(K\phi)(f)(x)=\int K(x,y)\phi(y)f(y)\mathrm{d}y.$$

Prove that $K\mathbb{1}_A$ is trace-class with spectrum in $[0,1]$. Let $X=\chi(A)$. Show that

$$\log\mathbb{E}(e^{\mathrm{i}\xi X})=\log\det(\mathrm{Id}+K\mathbb{1}_A(e^{\mathrm{i}\xi}-1))=-\sum_{k=1}^\infty\frac{(1-e^{\mathrm{i}\xi})^k}{k}\mathrm{Tr}((K\mathbb{1}_A)^k).$$

(v) The formula $\log\mathbb{E}(e^{\mathrm{i}\xi X})=\sum_{\ell=1}^\infty\mathrm{C}_\ell(X)\frac{(\mathrm{i}\xi)^\ell}{\ell!}$ defines the cumulants $\mathrm{C}_\ell(X)$ of the random variable X. Prove that for any $\ell\geq 3$,

$$\mathrm{C}_\ell(X)=(-1)^\ell(\ell-1)!\,\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^\ell)+\sum_{j=2}^{\ell-1}\alpha_{j\ell}\ \mathrm{C}_j(X)$$

for some universal constants $\alpha_{j\ell}$.

(vi) Take $A=[0,x)$ $(x\in(0,2\pi))$ in this question and the next one. Prove that

$$\mathrm{C}_2(X)=\int_0^x\mathrm{d}u\int_x^{2\pi}\mathrm{d}v\ |K(u,v)|^2\underset{N\to\infty}{\sim}\pi^{-2}\log N.$$

(vii) Prove that $\mathrm{C}_\ell(X/\sqrt{\log N})$ converges to 0 as $N\to\infty$ for any $\ell\geq 3$. For this you can first prove the trace inequality

$$0\leq\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^\ell)\leq(\ell-1)\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^2).$$

Show that $(X-\mathbb{E}\,X)/\sqrt{\log N}$ converges weakly to a Gaussian random variable with variance π^{-2}. Compare this result to the case of N independent uniform points on the circle.

(viii) Consider $X_k=\chi([0,x_k))-Nx_k/(2\pi)$ where $x_k=N^{-\alpha_k}$, $0<\alpha_1<\dots<\alpha_\ell<1$. Prove a joint central limit theorem for the random variables $X_1,\dots,X_\ell$ as $N\to\infty$. Compare this result to the case of N independent uniform points on the circle.

Problem 2: large deviations for the largest eigenvalue of Gaussian β-ensembles. In this problem, we consider the ordered eigenvalues of symmetric ($\beta = 1$) or Hermitian ($\beta = 2$) Gaussian matrices, i.e. the probability measure

$$\mathrm{d}\mathbb{P}_N(\lambda_1,\dots,\lambda_N) = \widetilde{Z}_N^{-1}|\Delta(\lambda_1,\dots,\lambda_N)|^\beta e^{-N\frac{\beta}{4}\sum_{i=1}^N \lambda_i^2}\mathbb{1}_{\lambda_1\leq\dots\leq\lambda_N}\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_N.$$

Let $\varrho(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$. We want to prove that λ_N satisfies a large deviations principle with good rate function

$$I(x) = \begin{cases} -\beta\int\log|x-y|\varrho(y)\mathrm{d}y + \frac{\beta}{4}x^2 - \frac{\beta}{2} & \text{if } x\geq 2 \\ \infty & \text{if } x<2 \end{cases}.$$

(i) Give a closed form for $\widetilde{Z}_N$.

(ii) Show that

$$\mathbb{P}_N(\lambda_N > L) \leq \frac{\widetilde{Z}_{N-1}}{\widetilde{Z}_N}\int_L^\infty e^{-N\frac{\beta}{4}\lambda_N^2}\mathrm{d}\lambda_N \int\prod_{i=1}^{N-1}\left(|\lambda_N-\lambda_i|^\beta e^{-\frac{\beta}{4}\lambda_i^2}\right)\mathrm{d}\mathbb{P}_{N-1}(\lambda_1,\dots,\lambda_{N-1}).$$

Using $|x-\lambda_i|e^{-\frac{\lambda_i^2}{4}}\leq Ce^{\frac{x^2}{8}}$, conclude that

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N(\lambda_N > L) = -\infty.$$

(iii) Let $\mathrm{d}\mathbb{P}'_N(\lambda_1,\dots,\lambda_N) = \widetilde{Z'_N}^{-1}|\Delta(\lambda_1,\dots,\lambda_N)|^\beta e^{-(N+1)\frac{\beta}{4}\sum_{i=1}^N\lambda_i^2}\mathbb{1}_{\lambda_1\leq\dots\leq\lambda_N}\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_N$. Moreover, denote $\mathscr{B}(\varepsilon)$ the ball of radius ε around ϱ, for the distance $\mathrm{d}(\mu,\nu) = \sup_{\|f\|_{\mathrm{Lip}}\leq 1}|\int f\mathrm{d}\mu - \int f\mathrm{d}\nu|$ (or any distance which metrizes the weak topology). Let $\mathscr{B}_L(\varepsilon)$ denote measures in $\mathscr{B}(\varepsilon)$ supported in $[-L,L]$. Prove that

$$\mathbb{P}_N(x<\lambda_N<L) \leq \mathbb{P}_N(\lambda_1<-L) + \frac{\widetilde{Z}'_{N-1}}{\widetilde{Z}_N}\left(\int_x^L e^{(N-1)\sup_{\mu\in\mathscr{B}_L(\varepsilon)}(\beta\int\log|\lambda-y|\mathrm{d}\mu(y) - \frac{\beta}{4}\lambda^2)}\mathrm{d}\lambda \right.$$
$$\left. +(L-x)e^{(N-1)\beta\log(2L)}\mathbb{P}'_{N-1}\left(\frac{1}{N-1}\sum_{k=1}^{N-1}\delta_{\lambda_k}\notin\mathscr{B}(\varepsilon)\right)\right)$$

Remember the large deviations principle for the empirical spectral measure. Prove that for any $x>2$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N(\lambda_N\geq x)\leq -I(x).$$

(iv) Let $x>2$, and $r,\varepsilon>0$ such that $2<r<x-2\epsilon$. Prove that

$$\mathbb{P}_N(x-\varepsilon<\lambda_N<x+\varepsilon) \geq \frac{\widetilde{Z}'_{N-1}}{\widetilde{Z}_N}\int_{x-\varepsilon}^{x+\varepsilon}\mathrm{d}\lambda_N\int_{[-L,r]^{N-1}} e^{\beta\sum_{k=1}^{N-1}\log|\lambda_N-\lambda_k| - N\frac{\beta}{4}\lambda_N^2}\mathrm{d}\mathbb{P}'_{N-1}(\mathrm{d}\lambda_1,\dots,\lambda_{N-1}).$$

Show that for any $\varepsilon>0$ and $x>2$ we have

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N(x-\varepsilon<\lambda_N<x+\varepsilon)\geq -I(x).$$

(v) Conclude the proof of the large deviations principle for λ_N.

(vi) Give (with no proof) a large deviations principle for the distribution of $(\lambda_{N-k},\dots,\lambda_N)$, where $k\geq 1$ is fixed.

Exercise 1. Fluctuations for the Ginibre ensemble. Consider the joint distribution of eigenvalues from the Ginibre ensemble,

$$\mathbb{P}(\mathrm{d}\boldsymbol{z}) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} |z_i - z_j|^2 \prod_{i=1}^{N} e^{-N|z_i|^2} \mathrm{dA}(z_i)$$

where dA is the Lebesgue measure on $\mathbb{C}$. Let $\mathscr{C}$ be a smooth Jordan curve, with interior A, finite length $\ell(\mathscr{C})$, strictly included in the unit disk $\{|z| < 1\}$. Let $X_{\mathscr{C}} = \chi(A) - \mathbb{E}(\chi(A))$ where $\chi = \sum_{i=1}^{N} \delta_{z_i}$. By mimicking the method from Problem 1, prove the weak convergence

$$\frac{X_{\mathscr{C}}}{\ell(\mathscr{C})^{1/2} N^{1/4}} \to \mathscr{N}(0, c)$$

as $N \to \infty$, with some c independent of $\mathscr{C}$. What about joint convergence of $(X_{\mathscr{C}_1}, \dots, X_{\mathscr{C}_n})$ where all Jordan curves $\mathscr{C}_1, \dots, \mathscr{C}_n$ satisfy the above assumptions?

Exercise 2. The semicircle law for band matrices. Let H_N be a symmetric matrix with $H_N(i,j)$ a standard Bernoulli random variable when $|i-j| \le W/2$ or $||i-j| - N| \le W/2$, 0 otherwise. All entries are independent, up to the symmetry constraint. Assume $1 \ll W \le N$.

Prove that the empirical spectral measure of $W^{-1/2} H_N$ converges (in probability, say) to the semicircle distribution $\varrho(s) = (2\pi)^{-1} \sqrt{(4 - s^2)_+}$.