Stochastic calculus, practice for the midterm exam

These are just examples of typical exercises for the midterm. You may also be asked to state important theorems and give some short proofs of results seen in class. Lecture notes will not be allowed.

Exercise 1. Let Y_1, Y_2, Y_3 be independent $\mathscr{N}(0,1)$ random variables. Let

$$X_1 = Y_1 + Y_3,\ X_2 = Y_2 + 4Y_3,\ X_3 = 2Y_1 - 2Y_2 + xY_3.$$

for some real number x.

(i) Explain why $\mathbf{X} = (X_1, X_2, X_3)$ is a Gausssian vector.
(ii) What is the covariance matrix for $\mathbf{X}$?
(iii) For what values of x are X_1 and X_3 independent?

Exercise 2. Let $X_1, X_2, \dots$ be independent random variables and and $\mathbb{P}(X_j = 2) = 1 - \mathbb{P}(X_j = -1) = \frac{1}{3}$ for any j. Let $S_n = X_1 + \cdots + X_n$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(i) Calculate $\mathbb{E}(S_n)$, $\mathbb{E}(S_n^2)$, $\mathbb{E}(S_n^3)$.
(ii) If $m < n$, calculate $\mathbb{E}(S_n \mid \mathcal{F}_m)$, $\mathbb{E}(S_n^2 \mid \mathcal{F}_m)$, $\mathbb{E}(S_n^3 \mid \mathcal{F}_m)$.
(iii) If $m < n$, calculate $\mathbb{E}(X_m \mid S_n)$.

Exercise 3. Let $X_1, X_2, \dots$ be independent random variables and and $\mathbb{P}(X_j = 1) = 1 - \mathbb{P}(X_j = -1) = \frac{1}{2}$ for any j. Let $S_n = X_1 + \cdots + X_n$. What is $\mathbb{E}(\sin S_n \mid S_n^2)$?

Exercise 4. Let $1/2 < q < 1$ and $X_1, X_2, \dots$ be independent random variables and and $\mathbb{P}(X_j = 1) = 1 - \mathbb{P}(X_j = -1) = q$ for any j. Let $S_n = X_1 + \cdots + X_n$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(i) Is S a $(\mathcal{F}_n)_{n\geq 0}$-martingale, submartingale, supermartingale?
(ii) Find r such that $S_n - rn$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.
(iii) Let $\theta = (1-q)/q$ and $M_n = \theta^{S_n}$. Prove that M a $(\mathcal{F}_n)_{n\geq 0}$-martingale.
(iv) For positive integers a and b, let $T_{a,b} = \inf\{n \geq 0 : S_n = a \text{ or } b\}$. Calculate $\mathbb{P}(S_{T_{a,b}} = b)$, under the assumption $S_0 \in (a, b)$.
(v) Let $T_a = T_{a,\infty}$. Find $\mathbb{P}(T_a < \infty)$.

Exercise 5. Let B be a standard Brownian motion, $\sigma > 0$ and $M_t = e^{\sigma B_t - \frac{\sigma^2}{2} t}$. Show that M is a martingale (with respect to which filtration?).

Exercise 6. Let B be a standard Brownian motion. Compute the following.

(1) $\mathbb{E}\left(B_t^2 \mid \mathcal{F}_s\right)$
(2) $\mathbb{E}\left(B_t^3 \mid \mathcal{F}_s\right)$
(3) $\mathbb{E}\left(B_t^4 \mid \mathcal{F}_s\right)$
(4) $\mathbb{E}\left(e^{4B_t + 12} \mid \mathcal{F}_s\right)$

Exercise 7. Let B be a standard Brownian motion and $M_t = \max_{0\leq s\leq t} B_s$.

(1) Explain why M_t has the same distribution as $\sqrt{t} M_1$.
(2) What is the density of M_t?

Exercise 8. Let B be a standard Brownian motion and $\lambda > 0$. What is $\mathbb{E}(e^{-\lambda B_t})$? What is $\mathbb{E}(e^{-\lambda B_t^2})$?

Exercise 9. Let B be a standard Brownian motion Show that $\int_0^1 \frac{B_s}{s} \mathrm{d}s$ converges almost surely.

Exercise 10. Let B be a standard Brownian motion and $T_1 = \inf\{s \geq 0 : B_s = 1\}$. What is the distribution of $-\inf_{s \leq T_1} B_s$?

Exercise 11. Let M be a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that $M_t \in \mathrm{L}^2$ for any t. Prove that

$$\mathbb{E}\left(M_t^2 - M_s^2 \mid \mathcal{F}_s\right) = \mathbb{E}\left((M_t - M_s)^2 \mid \mathcal{F}_s\right).$$

Exercise 12. Let B be a standard Brownian motion. For any $t > 0$ and each positive integer n, let

$$\Delta_t^n = \sum_{j=1}^{n} \left(B_{\frac{j}{n}t} - B_{\frac{j-1}{n}t}\right)^2.$$

(i) Find the mean and variance of the random variable Δ_t^n.
(ii) Prove the following convergence: for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(|\Delta_t^n - t| > \varepsilon\right) = 0.$$