Exercise 1. Let \( B \) be a Brownian motion and \( T_a = \inf\{ s \geq 0 : B_s = a \} \). Prove (with no calculation) that for any \( b > a > 0 \) the random variable \( T_b - T_a \) is independent of \( T_a \).

Simulate a Brownian motion (based on the method suggested by Donsker's theorem, with a thousand time steps between times 0 and 1) and the corresponding process \( (T_a)_{a \geq 0} \). Give a printed copy of the code with the homework (no matter which language) and a samples of both curves.

Exercise 2. Let \( B \) be a Brownian motion starting at \( x > 0 \), and \( T_0 = \inf\{ s \geq 0 : B_s = 0 \} \). That is the distribution of \( \sup_{t \leq T_0} B_t \)?

Hint: at some point in class we studied maxima of positive martingales converging to 0.

Exercise 3. Let \( X_t = \int_0^t (\sin s) dB_s \). Prove that this is a Gaussian process. What are \( \mathbb{E}(X_t) \) and \( \mathbb{E}(X_s X_t) \)? Prove that

\[
X_t = (\sin t) B_t - \int_0^t (\cos s) B_s ds.
\]

Exercise 4. Prove that if \( f \) is a deterministic continuous square integrable function,

\[
\mathbb{E} \left( B_t \int_0^\infty f(s) dB_s \right) = \int_0^t f(s) ds.
\]