Stochastic calculus, homework 9, due December 4th.

Below B, $B^{(1)}$, $B^{(2)}$ ... are standard, independent, Brownian motions.

Exercise 1. Prove that the following processes are martingales with respect to the Brownian filtration.

(i) $X_t = e^{\frac{t}{2}} \cos(B_t)$;

(ii) $X_t = (B_t + t)e^{-B_t - \frac{t}{2}}$.

Exercise 2. Let X be a stochastic proces starting at X_0 and satisfying $\mathrm{d}X_t = (-\alpha X_t + \beta)\mathrm{d}t + \sigma \mathrm{d}B_t$, where $\alpha > 0$.

(i) Give an explicit expression for X_t.

(ii) Calculate $\mathrm{Cov}(X_s, X_t)$ for any $s < t$.

Exercise 3. Let $M^{(1)}, \dots, M^{(d)}$ be local martingales starting at 0.

(i) Prove that the following assertions are equivalent:

  (i) The processes $M^{(1)}, \dots, M^{(d)}$ are independent standard Brownian motions.

  (ii) For any $1 \leq k, \ell \leq d$ and $t \geq 0$, $\langle M^{(k)}, M^{(\ell)} \rangle_t = \mathbb{1}_{k=\ell} t$.

Hint: find, understand and rewrite the proof from the lecture notes.

(ii) As an example, consider a Brownian motion B, and define the process $\tilde{B}$ through

$$\tilde{B}_t = \int_0^t \mathrm{sgn}(B_s) \mathrm{d}B_s,$$

where $\mathrm{sgn}(x) = 1$ if $x \geq 0$, -1 if $x < 0$. Prove that $\tilde{B}$ is a Brownian motion. Is it almost surely equal to B?

Exercise 4. Assume the process $(S_t)_{t \geq 0}$ satisfies $\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma \mathrm{d}B_t)$ for some Brownian motion B which corresponds to the Wiener measure $\mathbb{P}$. Let $Z_t = e^{\frac{1}{t}\int_0^t \log S_s \mathrm{d}s}$. We want to calculate $C = \mathbb{E}_{\mathbb{P}}((Z_t - S_t)_+)$

(i) Prove that if we define $\mathrm{d}\mathbb{Q} = e^{\sigma B_t - \sigma^2 \frac{t}{2}} \mathrm{d}\mathbb{P}$ then

$$e^{-rt} \mathbb{E}_{\mathbb{P}}((Z_t - S_t)_+) = \mathbb{E}_{\mathbb{Q}}((\frac{Z_t}{S_t} - 1)_+).$$

(ii) Let $\widetilde{B}_t = B_t - \sigma t$. Write $\frac{Z_t}{S_t}$ as $e^{\alpha t - \int_0^t \beta_s \mathrm{d}\widetilde{B}_s}$.

(iii) Explain how to calculate C from the knowledge of $\mathbb{E}((\widetilde{S}_t - K)_+)$ where$\widetilde{S}_t$ ius an explict geometric Brownian motion.

Exercise 5. Assume the process $(X_t)_{t \geq 0}$ satisfies $\mathrm{d}X_t = X_t(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t)$ for some Brownian motion B which corresponds to the Wiener measure $\mathbb{P}$.

(i) Prove that $X_t e^{-\int_0^t \mu_s \mathrm{d}s}$ is a local martingale under $\mathbb{P}$.

(ii) Find a probability $\mathbb{Q}$ under which X is a local martingale.

(iii) Find a probability $\widetilde{\mathbb{Q}}$ under which X^{-1} is a local martingale.

Exercise 6. *(bonus)* Let B be a Brownian motion, $a > 0$, $\gamma \in \mathbb{R}$, and $S_{a,\gamma} = \inf\{t \geq 0 \mid |B_t + \gamma t| = a\}$. Are $S_{a,\gamma}$ and $B_{S_{a,\gamma}} + \gamma S_{a,\gamma}$ independent?