Lecture notes are not be allowed. Exercises with a (∗) are probably more difficult than the others.

**Exercise 1.** Define what a submartingales in discrete time is.

**Exercise 2.** Let $(X_n)_{n \geq 1}$ be a martingale in discrete time and $S \leq T$ be two stopping times. Give an example in which $\mathbb{E}(X_T | \mathcal{F}_S) = X_S$ does not hold (no need for a proof).

**Exercise 3.** Let $X$ be a nonnegative martingale in discrete time. Does it converge almost surely? If yes, thanks to which theorem? If no, give a counterexample (no need for a proof).

**Exercise 4.** Given a filtration in discrete time, define a stopping time. Give an example. Give an example of a time which is not a stopping time (no need for a proof).

**Exercise 5.** Let $X$ be a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. For any real $t$, what is $\mathbb{E}(e^{tX})$?

**Exercise 6.** Let $X$ be a standard Gaussian random variable, and $\varepsilon$ an independent Bernoulli random variable ($\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$). Is $(X, \varepsilon X)$ a Gaussian vector? Justify your answer.

**Exercise 7.** Define a standard Brownian motion.

**Exercise 8.** Let $B$ be a standard Brownian motion and $\lambda > 0$. Is $(\sqrt{\lambda}B_{rt})_{t \geq 0}$ a standard Brownian motion? Justify your answer.

**Exercise 9.** Let $B$ be a standard Brownian motion. For which $\alpha > 0$ does $\lim_{t \to 0^+} \frac{B_t}{t^\alpha}$ exist almost surely? Then, what is the limit? You don’t need to prove anything, just answer.

**Exercise 10.** Let $B$ be a standard Brownian motion. Are the following assertions right or wrong? If right, give the theorem it relies on. If wrong, give a counterexample (no proof needed, just a counterexample).

(i) If $T$ is an almost surely finite stopping time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a standard Brownian motion.

(ii) If $T$ is an almost surely finite random time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a standard Brownian motion.

**Exercise 11.** Let $(X_k)_{k \geq 0}$ be i.i.d. standard Gaussian random variables, $\mathcal{F}_n = \sigma(X_k, k \leq n)$, and $S_n^{(\mu)} = \sum_{k=0}^{n} X_k + \mu n$. For which values of the real parameter
Exercise 12. With the same notations as in exercise 11, find $c > 0$ such that $\{(S_n^{(0)})^2 - cn\}_{n \geq 0}$ is a $(\mathcal{F}_n)_{n \geq 0}$-martingale. Prove it.

Exercise 13. With the same notations as in exercise 11, find $c > 0$ such that $(e^{S_n^{(0)} - cn})_{n \geq 0}$ is a $(\mathcal{F}_n)_{n \geq 0}$-martingale. Prove it. What does this martingale converge to, almost surely? Prove it.

Exercise 14 (*). State Donsker’s theorem. With the same notations as in exercise 11, find $\alpha > 0$ such that $\sum_{k \leq n} S_k^{(0)}$ converges as $n \to \infty$, to a non-trivial distribution. What is the density of this limit?

Exercise 15. Let $B$ be a standard Brownian motion. Calculate $\mathbb{E}\left(\int_0^1 B_s^2 ds\right)$.

Exercise 16 (*). Let $B$ be a standard Brownian motion, $\mu \in \mathbb{R}$. Calculate $\mathbb{E}\left(\sup_{0 \leq s \leq 1} e^{\mu B_s}\right)$.

Exercise 17. Let $B$ be a standard Brownian motion. What is $\lim_{t \to \infty} \frac{B_t}{t}$, almost surely? Prove it.

Exercise 18. Let $X$ be a $\mathcal{F}$-measurable integrable random variable, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}$ be a filtration. Prove that $(\mathbb{E}(X \mid \mathcal{F}_n))_{n \geq 1}$ is a $(\mathcal{F}_n)_{n \geq 1}$-martingale.

Exercise 19 (*). Let $B$ be a standard Brownian motion and $T_1^* = \inf\{t \geq 0 \mid |B_t| = 1\}$. Prove that $\sup_{0 \leq t \leq 1} |B_t|$ and $\frac{1}{\sqrt{T_1^*}}$ have the same distribution.

Exercise 20 (*). Let $B^{(1)}$ and $B^{(2)}$ be independent standard Brownian motions. Define the complex-valued process $B_t = B^{(1)}_t + iB^{(2)}_t$. Let $D$ be a straight line in the complex plane. What is the distribution of $T = \inf\{t \geq 0 : B_t \in D\}$?