Ewens measures on compact groups and hypergeometric kernels

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Summary. On unitary compact groups the decomposition of a generic element into product of reflections induces a decomposition of the characteristic polynomial into a product of factors. When the group is equipped with the Haar probability measure, these factors become independent random variables with explicit distributions. Beyond the known results on the orthogonal and unitary groups ($O(n)$ and $U(n)$), we treat the symplectic case. In $U(n)$, this induces a family of probability changes analogous to the biassing in the Ewens sampling formula known for the symmetric group. Then we study the spectral properties of these measures, connected to the pure Fisher-Hartvig symbol on the unit circle. The associated orthogonal polynomials give rise, as n tends to infinity to a limit kernel at the singularity.

Key words: Decomposition of Haar Measure, Random Matrices, Characteristic Polynomials, Ewens sampling formula, correlation kernel.

1 Introduction

In this paper, $U(n, K)$ is the unitary group over $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the set of real quaternions).

Let U be distributed with the Haar measure on $U(n, \mathbb{C})$. The random variable $\det(\mathrm{Id}_n - U)$ has played a crucial role in recent years in the study of some connections between random matrix theory and analytic number theory (see [21] for more details). In [10], the authors show that $\det(\mathrm{Id}_n - U)$ can be decomposed as a product of n independent random variables:

$$\det(\mathrm{Id}_n - U) \stackrel{\text{law}}{=} \prod_{k=1}^{n} \left(1 - e^{i\omega_k} \sqrt{B_{1,k-1}}\right), \tag{1}$$

where $\omega_1, \dots, \omega_n, B_{1,0}, \dots, B_{1,n-1}$ are independent, the $\omega_k's$ being uniformly distributed on $(-\pi, \pi)$ and the $B_{1,j}$'s $(0 \leq j \leq n-1)$ being beta distributed with parameters 1 and j (with the convention that $B_{1,0} = 1$). In particular, from such a decomposition, fundamental quantities such as the Mellin-Fourier transform of $\det(\mathrm{Id}_n - U)$ follow at once. The main ingredient to obtain the decomposition (1) is a recursive construction of the Haar measure using complex reflections. In particular, every $U \in U(n, \mathbb{C})$ can be decomposed as a product of n independent reflections. More precisely, it is proved in [10] that if $s_1, \dots, s_n$ are n independent random variables such that for every $k \leq n$, s_k is uniformly distributed on the k-th dimensional unit sphere $\mathscr{S}^k$ in $\mathbb{C}^k$ and if $R^{(k)}$ is the reflection of $\mathbb{C}^k$ mapping s_k onto the first vector of the canonical basis, then

$$R^{(n)} \begin{pmatrix} \mathrm{Id}_1 & 0 \\ 0 & R^{(n-1)} \end{pmatrix} \dots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & R^{(2)} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & R^{(1)} \end{pmatrix} \sim \mu_{U(n,\mathbb{C})},$$

where $\mu_{U(n,\mathbb{C})}$ stands for the Haar measure on $U(n, \mathbb{C})$. At this stage two remarks are in order. First, a similar method works to generate the Haar measure on the orthogonal group $O(n, \mathbb{R})$ (see [10]) and this was already noticed by Mezzadri in [25] using Householder reflections. But as already noticed in [10], Householder reflections would not work for $U(n, \mathbb{C})$ (see next section for more details). Moreover in [10], a decomposition such as (1) could not be obtained for the symplectic group $\mathrm{USp}(2n, \mathbb{C})$, which also plays an important role in the connections between random matrix theory and the study of families of L functions (see [19], [20]). Indeed, there does not seem to be a natural way to generate recursively the Haar measure on this group.

Question 1. Is there any decomposition of $\det(\mathrm{Id}_n - U)$ as a product of independent variables of the type (1), when U is drawn from $\mathrm{USp}(2n, \mathbb{C})$, according to the Haar measure?

In this paper we shall prove that, in a sense to be made precise, if a subgroup $\mathcal{G}$ of $U(n, K)$ contains enough reflections, then one can recursively generate the Haar measure and obtain a decomposition of the type (1) for $\det(\mathrm{Id}_n - U)$, $U \in \mathcal{G}$. In particular this will apply to $U(n, \mathbb{H})$ which can be identified with the symplectic group, hence answering question 1 above. Our recursive decomposition of the Haar measure also applies to the symmetric group. This leads us to our second remark concerning the generation of the Haar measure obtained in [10] and explained above. Indeed, this way of generating an element of $U(n, \mathbb{C})$ which is Haar distributed by choosing a vector $(s_1, \dots, s_n)$ of independent variables from $\mathscr{S}^1 \times \dots \times \mathscr{S}^n$, each s_i being uniformly distributed, is reminiscent of the generation of a random permutation according to the so-called Chinese restaurant process which we briefly describe (see [32] for a complete treatment). Let $[n]$ denote the set $\{1, \cdots, n\}$ and $\mathcal{S}_n$ the symmetric group of order n. It is known that for $n \geq 2$, every permutation $\sigma \in \mathcal{S}_n$ can be decomposed in the following way:

$$\sigma = \tau_n \circ \cdots \circ \tau_2 \tag{2}$$

where for $k = 2, \ldots, n$, either τ_k is the identity or τ_k is the transposition (k, m_k) for some $m_k \in [k-1]$. In the first case we will say by extension that it is the transposition (k, m_k) with $m_k = k$. This decomposition is unique, see Tsilevich [36], the lemma p. 4075. It corresponds to the Chinese restaurant generation of a permutation. Let us consider cycles as "tables". Integer 1 goes to the first table. If $\tau_2 \neq \mathrm{Id}$, then integer 2 goes to the first table, at the left of 1. If $\tau_2 = \mathrm{Id}$, it goes to a new table. When integers $1, \ldots, k$ are placed, then $k+1$ goes to a new table if $\tau_{k+1} = \mathrm{Id}$, and goes to the left of $\tau_{k+1}(k+1) = j_{k+1}$ if not. We get a bijection between $[1] \times [2] \times \cdots \times [n] \to \mathcal{S}_n$. It is projective (or consistent) in the sense that if σ is in $\mathcal{S}_{n+1}$ the restriction of σ to $[n]$ is in $\mathcal{S}_n$.

In this setting, the number of cycles k_σ of a permutation σ is the number of tables, i.e. the number of Id in (2) i.e. [1]

$$k_\sigma = \sum_1^n \xi_r\,, \tag{3}$$

where $\xi_r = 1(\tau_r = \mathrm{Id})$. For a matricial rewriting, we make a change of basis. Let $e'_j = e_{n-j+1}$ and let $R^{(k)}$ be the restriction of τ_k to $[k]$. Then the product in (2) is represented by

$$R^{(n)} \begin{pmatrix} \mathrm{Id}_1 & 0 \\ 0 & R^{(n-1)} \end{pmatrix} \cdots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & R^{(2)} \end{pmatrix}.$$

If at each stage, the integer m_k is chosen uniformly in $[k]$, then the induced measure on $\mathcal{S}_n$ is the uniform distribution denoted by $\mu_{\mathcal{S}_n}$.

Actually, one can more generally generate in this way the Ewens measure on $\mathcal{S}_n$ (see Tsilevich [36] and Pitman [32]). The Ewens measure $\mu^{(\theta)}$, $\theta > 0$, is a deformation of $\mu_{\mathcal{S}_n}$ obtained by performing a change of probability measure or a sampling in the following way:

$$\mu_n^\theta(\sigma) = \frac{\theta^{k_\sigma}}{(\theta)_n} \cdot \mu_{\mathcal{S}_n}(\sigma)\,. \tag{4}$$

To generate μ_n^θ, one has to pick n integers $m_1, m_2, \ldots, m_n$, independently, from $[1] \times \cdots \times [n]$ according to the probability distribution

$$\mathbb{P}(m_k = k) = \frac{\theta}{\theta + k - 1}\ ,\ \mathbb{P}(m_k = j) = \frac{1}{\theta + k - 1}\ j = 1, \cdots, k-1\,.$$

Question 2. Is there an analogue of the Ewens measure on the unitary group $U(n, \mathbb{C})$?

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[1] The other construction of a random permutation named Feller's coupling uses the variables in the reverse order $\xi_n, \cdots, \xi_1$, but this construction is not projective.

We shall see in this paper that there indeed exists an analogue of the Ewens measure on $U(n,\mathbb{C})$: more precisely we generalize (4) to unitary groups and a particular class of their subgroups. The analogue of transpositions are reflections and the weight of the sampling is now $\det(\mathrm{Id}-U)^{\overline{\delta}}\det(\mathrm{Id}-\overline{U})^{\delta}$, $\delta\in\mathbb{C}$, $\mathfrak{Re}(\delta)>-1/2$, so that the measure $\mu_{U(n)}^{(\delta)}$ on $U(n)$, which is defined by

$$\mathbb{E}_{\mu_{U(n)}^{(\delta)}}(f(U))=\frac{\mathbb{E}_{\mu_{U(n)}}\left(f(U)\det(\mathrm{Id}-U)^{\overline{\delta}}\det(\mathrm{Id}-\overline{U})^{\delta}\right)}{\mathbb{E}_{\mu_{U(n)}}\left(\det(\mathrm{Id}-U)^{\overline{\delta}}\det(\mathrm{Id}-\overline{U})^{\delta}\right)}$$

for any test function f, is the analogue of the Ewens measure. Such samplings with $\delta\in\mathbb{R}$ have already been studied on the finite-dimensional unitary group by Hua [18], and results about the infinite dimensional case (on complex Grassmannians) were given by Pickrell ([30] and [31]). More recently, Neretin [27] also considered this measure, introducing the possibility $\delta\in\mathbb{C}$. Borodin and Olshanski [7] have used the analogue of this measure in the framework of the infinite dimensional unitary group and proved ergodic properties. Forrester and Witte in [38] referred to this measure as the cJUE distribution. We also studied this ensemble in [12] in relation with the theory of orthogonal polynomials on the unit circle. Following [38] and [12] we shall call the ensemble of unitary matrices endowed with this sampled measure the *circular Jacobi ensemble*.

It is natural to ask whether the circular Jacobi ensemble has some interesting properties: indeed, the case $\delta=0$ corresponds to the Haar measure and it is well known this ensemble enjoys many remarkable spectral properties. For instance, the point process associated to the eigenvalues is determinantal and the associated rescaled kernel converges to the sine kernel . The projection of the measures $\mu_{U(n)}^{(\delta)}$ on the spectrum has the density

$$\frac{1}{\mathcal{Z}_n}\prod_{j=1}^{n}w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta_j})\prod_{1\le i<j\le n}|\mathrm{e}^{\mathrm{i}\theta_i}-\mathrm{e}^{\mathrm{i}\theta_j}|^2$$

where the weight $w^{\mathbb{T}}$ on $\mathbb{T}=\{\mathrm{e}^{\mathrm{i}\theta},\theta\in[-\pi,\pi]\}$ is defined by

$$w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})=(1-\mathrm{e}^{\mathrm{i}\theta})^{\overline{\delta}}(1-\mathrm{e}^{-\mathrm{i}\theta})^{\delta}=(2-2\cos\theta)^{a}\mathrm{e}^{-b(\pi\operatorname{sgn}\theta-\theta)}\,,$$

($\delta=a+\mathrm{i}b$) and $\mathcal{Z}_n$ is a normalization constant. Note that when $b\neq0$, an asymmetric singularity at 1 occurs. The statistical properties of the θ_k's depend on the successive orthonormal polynomials (φ_k) with respect to the normalized version $\widetilde{w}^{\mathbb{T}}$ of $w^{\mathbb{T}}$ and the normalized reproducing kernel

$$\widetilde{K}_n^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}\tau})=\sqrt{\widetilde{w}^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{w}^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\tau})}\sum_{\ell=0}^{n-1}\overline{\varphi_\ell(\mathrm{e}^{\mathrm{i}\theta})}\varphi_\ell(\mathrm{e}^{\mathrm{i}\tau})\,.$$

In [7] the authors consider the image of $\mu_{U(n)}^{(\delta)}$ by the Cayley transform on the set of Hermitian matrices and make a thorough study of the spectral properties of this random matrix ensemble. In particular they prove that the eigenvalues form a determinantal process and show that the associated rescaled kernel converges to some hypergeometric kernel. As expected, we shall see that the eigenvalues process of the circular Jacobi ensemble is also determinantal and for every n, we identify the hypergeometric kernel $K_n^{(\delta)}$ associated with it.

Question 3. Is there an appropriate rescaling of the kernels $K_n^{(\delta)}$ such that the rescaled kernels converge to some kernel $K_\infty^{(\delta)}$?

We shall see that the answer to question 3 is positive and that the kernel $K_\infty^{(\delta)}$ is a *confluent hypergeometric kernel*, with a natural connection to that obtained by Borodin and Olshanski in [7] on the set of Hermitian matrices. The case $\delta = 0$ corresponds to the sine kernel.

The weight $w^{\mathbb{T}}$ is a generic example leading to a singularity

$$c^{(+)}|\theta|^{2a}\mathbb{1}_{\theta>0} + c^{(-)}|\theta|^{2a}\mathbb{1}_{\theta<0}$$

at $\theta = 0$, with distinct positive constants $c^{(+)}$ and $c^{(-)}$. The confluent hypergeometric kernel, depending on the two parameters a and $b = \frac{1}{2\pi}\log(c^{(-)}/c^{(+)})$, is actually universal for the measures presenting the above singularity, as proved in a forthcoming paper, following the method initiated by Lubinsky ([22], [23]). For a universality result when δ is real see [33].

The layout of the paper is as follows. In Section 2 we present the generation by reflections and deduce a splitting formula for the characteristic polynomial (Theorem 2). As an application, we define the generalized Ewens measure depending on the complex parameter δ (Theorem 3). Section 3 is devoted to a study of the kernel which governs the correlations of eigenvalues when the unitary group is equipped with this measure and its asymptotics (Theorem 5). The main properties of the families of hypergeometric functions ${}_2F_1$ and ${}_1F_1$ are recalled in the Appendix.

2 Generating the Haar measure and the generalized Ewens measure

2.1 Complex reflections

Reflections play a central role in the generation of the Haar measure for the classical compact groups. In the case of $O(n)$ the decomposition into a product of reflections is well known, see [15] and other references as explained in [25]. Householder reflections are generally used in the case of $O(n)$, but they are not suitable for $U(n, \mathbb{C})$. Indeed, recall that Householder reflections are of the form $H_v = \mathrm{Id} - 2v\langle v|\cdot\rangle$. For every unit y, it is possible to choose v such that $H_v y = \alpha e_1$ with $\alpha = \pm\frac{y_1}{|y_1|}$, where e_1 is the first element of the canonical basis. So when the ground field is $\mathbb{C}$, then $\alpha \neq 1$ in general and there does not exist a Householder reflection which maps y onto e_1, whereas this can always be achieved when the ground field is $\mathbb{R}$. That is why it is not possible to directly extend the arguments in [25] to $U(n, \mathbb{C})$. In [10] and [12] it is proposed to use complex (resp. quaternionic) proper reflections, that is norm preserving automorphisms of $\mathbb{C}^n$ (resp. $\mathbb{H}^n$) that leave exactly one hyperplane pointwise fixed. So a reflection will be either the identity or a unitary transformation U such that $I - U$ is of rank one. It may be written as

$$s_{a,\lambda}(y) = y - a\frac{(1-\lambda)\langle a, y\rangle}{|a|^2}$$

where $a \in \mathbb{H}^n$ and $\lambda \in \mathbb{H}$ with $|\lambda| = 1$ (λ is the second eigenvalue). If $x \neq e_1$, there exists a reflection mapping e_1 onto x. It is enough to take $a = e_1 - x$ and $\lambda = -(1-x_1)(1-\bar{x}_1)^{-1}$ where $x_1 = \langle e_1, x\rangle$.

2.2 Generating the Haar measure on $U(n, K)$ and on some of its subgroups

We first give conditions under which an element of a subgroup of $U(n, K)$ (under the Haar measure) can be generated as a product of independent reflections. This will lead to some remarkable identities for the characteristic polynomial.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{K}^n$. Let $\mathcal{G}$ be a subgroup of $U(n, K)$ and for all $1 \leq k \leq n-1$, let

$$\mathcal{H}_k = \{G \in \mathcal{G} \mid G(e_j) = e_j,\ 1 \leq j \leq k\},$$

the subgroup of $\mathcal{G}$ which stabilizes $e_1, \cdots, e_k$. We set $\mathcal{H}_0 = \mathcal{G}$. For a generic compact group $\mathcal{A}$, we write $\mu_{\mathcal{A}}$ for the unique Haar probability measure on $\mathcal{A}$. Finally for all $1 \leq k \leq n$ let p_k be the map $U \mapsto U(e_k)$.

Proposition 1. *Let $G \in \mathcal{G}$ and $H \in \mathcal{H}_1$ be independent random matrices, and assume that $H \sim \mu_{\mathcal{H}_1}$. Then $GH \sim \mu_{\mathcal{G}}$ if and only if $G(e_1) \sim p_1(\mu_{\mathcal{G}})$.*

Proof. The proof is exactly the same as in [10] Prop. 2.1, changing $U(n+1)$ into $\mathcal{G}$ and $U(n)$ into $\mathcal{H}$.

Definition 1. *A sequence $(\nu_0, \ldots, \nu_{n-1})$ of probability measures on $\mathcal{G}$ is said to be coherent with $\mu_{\mathcal{G}}$ if for all $0 \leq k \leq n-1$,*

$$\nu_k(\mathcal{H}_k) = 1 \text{ and } p_{k+1}(\nu_k) = p_{k+1}(\mu_{\mathcal{H}_k}).$$

In the following, $\nu_0 \star \nu_1 \star \cdots \star \nu_{n-1}$ stands for the law of a random variable $H_0 H_1 \ldots H_{n-1}$ where all H_i's are independent and $H_i \sim \nu_i$. Now we can provide a general method to generate an element of $\mathcal{G}$ endowed with its Haar measure.

Theorem 1. *If $\mathcal{G}$ is a subgroup of $U(n,K)$ and $(\nu_0,\dots,\nu_{n-1})$ is a sequence of coherent measures with $\mu_{\mathcal{G}}$, then we have:*

$$\mu_{\mathcal{G}} = \nu_0 \star \nu_1 \star \cdots \star \nu_{n-1}.$$

Proof. It is sufficient to prove by induction on $1 \le k \le n$ that

$$\nu_{n-k} \star \nu_{n-k+1} \star \cdots \star \nu_{n-1} = \mu_{\mathcal{H}_{n-k}},$$

which gives the desired result for $k = n$. If $k = 1$ this is obvious. If the result is true at rank k, it remains true at rank $k+1$ by a direct application of Proposition 1 to the groups $\mathcal{H}_{n-k-1}$ and its subgroup $\mathcal{H}_{n-k}$.

As an example, take the orthogonal group $O(n)$. Let $\mathscr{S}_{\mathbb{R}}^{(k)}$ be the unit sphere $\{x \in \mathbb{R}^k \mid |x| = 1\}$ and, for $s_k \in \mathscr{S}_{\mathbb{R}}^{(k)}$, let $R^{(k)}$ be the matrix of the reflection which transforms s_k into e_1. If s_k is uniformly distributed on $\mathscr{S}_{\mathbb{R}}^{(k)}$ and if all the s_k are independent, then by Theorem 1, the matrix

$$R^{(n)} \begin{pmatrix} 1 & 0 \\ 0 & R^{(n-1)} \end{pmatrix} \cdots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & R^{(2)} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & R^{(1)} \end{pmatrix}.$$

is $\mu_{O(n)}$ distributed.

2.3 Splitting of the characteristic polynomial

In view to phrase a general version of formula (1) which is proved in [10], we need the following definition:

Definition 2. *Note $\mathcal{R}_k$ the set of elements in $\mathcal{H}_k$ which are reflections. If for all $0 \le k \le n-1$*

$$\{R(e_{k+1}) \mid R \in \mathcal{R}_k\} = \{H(e_{k+1}) \mid H \in \mathcal{H}_k\},$$

the group $\mathcal{G}$ will be said to satisfy condition (R) (R standing for reflection).

Remark 1. It is easy to see that $U(n,K)$ and $\mathcal{S}_n$ satisfy condition (R). In the next subsection we shall see more examples.

Lemma 1. *Let $\mathcal{G}$ be a subgroup of $U(n,K)$ which satisfies condition (R). Let $G \in \mathcal{G}$. Then there exist reflections $R_k \in \mathcal{R}_k$, $0 \le k \le n-1$, such that*

$$G = R_0 R_1 \dots R_{n-1}. \tag{5}$$

Proof. This result has been established in [12] when $\mathcal{G} = U(n,\mathbb{C})$. The proof in this more general case goes exactly along the same line.

The following deterministic lemma is a key result to obtain a decomposition of $\det(\mathrm{Id}_n - U)$ as a product of independent random variables:

Lemma 2. *If for $k = 1, \dots, n-1$, $R_k \in \mathcal{R}_k$, then*

$$\det(\mathrm{Id}_n - R_0 \cdots R_{n-1}) = \prod_{k=0}^{n-1} \left(1 - \langle e_{k+1}, R_k(e_{k+1}) \rangle\right) . \tag{6}$$

Proof. We start with $\det(\mathrm{Id}_n - RH) = (\det H) \det(H^* - R)$. Since H (hence H^*), stabilizes e_1, we have

i) $(H^* - R)(e_1) = e_1 - R(e_1) =: a$ (say),

ii) for $w \perp e_1$, $H^*(w) \perp e_1$ and since R is a reflection, $R(w) - w$ is a scalar multiple of a.

By the multilinearity of the determinant, we get

$$\det(H^* - R) = \langle e_1, e_1 - R(e_1) \rangle \det(\pi(H^*) - \mathrm{Id}_{n-1})$$

which yields

$$\det(\mathrm{Id}_n - RH) = (1 - \langle e_1, R(e_1) \rangle) \det(\mathrm{Id}_{n-1} - \pi(H)) .$$

Iterating, we can conclude. □

The following result now follows immediately from Theorem 1 and Lemmas 1 and 2.

Theorem 2. *Let $\mathcal{G}$ be a subgroup of $U(n, K)$ satisfying condition (R), and let $(\nu_0, \dots, \nu_{n-1})$ be coherent with $\mu_{\mathcal{G}}$. If $G \sim \mu_{\mathcal{G}}$, then*

$$\det(\mathrm{Id} - G) \overset{\text{law}}{=} \prod_{k=0}^{n-1} \left(1 - \langle e_{k+1}, H_k(e_{k+1}), \rangle\right) .$$

where $H_k \sim \nu_k$, $0 \le k \le n-1$, are independent.

2.4 Applications

The symmetric group.

Consider now $\mathcal{S}_n$ the group of permutations of size n. An element $\sigma \in \mathcal{S}_n$ can be identified with the matrix $(\delta^j_{\sigma(i)})_{1 \le i,j \le n}$ (δ is Kronecker's symbol). It is clear that 1 is eigenvalue of this matrix, with eigenvector $e_1 + \dots + e_n$. Ben Hambly et al. [17] considered the characteristic polynomial at $s \neq 1$. To make relevant our problem of determinant splitting, we introduce wreath products, following the definition of Wieand [37].

Let F be a subgroup of $\mathbb{T} = \left\{x \in \mathbb{C} \mid |x|^2 = 1\right\}$, endowed with the Haar probability measure μ_F. Then the wreath product $F \wr \mathcal{S}_n$ provides another example of determinant-splitting. An element of F^n can be thought of as a function from the set $[n]$ to F. The group $\mathcal{S}_n$ acts on F^n in the following way: if $f = (f(1), \dots, f(n)) \in F^n$ and $\sigma \in \mathcal{S}_n$, define $f_\sigma \in F^n$ to be the function $f_\sigma = f \circ \sigma^{-1}$. Finally take the product on F^n to be

$(f(1),\dots,f(n))\cdot(g(1),\dots,g(n)) = (fg(1),\dots,fg(n))$. The wreath product of F by $\mathcal{S}_n$, denoted $F \wr \mathcal{S}_n$, is the group of elements $\{(f;s) : f \in F^n, \sigma \in \mathcal{S}_n\}$ with multiplication

$$(f;\sigma)\cdot(h;\sigma') = (fh_\sigma;\sigma\sigma')\,.$$

If we represent $(f;\sigma)$ by the matrix $(f(i)\delta^i_{\sigma(j)})_{1\le i,j\le n}$, then the product in $F\wr\mathcal{S}_n$ corresponds to the usual matricial product which makes $F\wr\mathcal{S}_n$ a subgroup of $U(n,\mathbb{C})$. The usual examples are $F=\{1\}$, $F=\mathbb{Z}_2$ and $F=\mathbb{T}$.

Corollary 1. *Let $G\in\mathcal{G}(=F\wr\mathcal{S}_n)$ be $\mu_{\mathcal{G}}$ distributed. Then*

$$\det(\mathrm{Id}_n - G) \overset{\text{law}}{=} \prod_{j=1}^n (1-\varepsilon_j X_j)\,,$$

with $\varepsilon_1,\dots,\varepsilon_n,X_1,\dots,X_n$ independent random variables, the ε_j's μ_F distributed, $\mathbb{P}(X_j=j)=1/j$, $\mathbb{P}(X_j=0)=1-1/j$.

Proof. We apply Theorem 2. As reflections correspond now to transpositions, condition (R) holds. Moreover $R_k(e_{k+1})$ is uniformly distributed on the set $Fe_{k+1}\cup\dots\cup Fe_n$, so that $\langle e_{k+1}, R_k(e_{k+1})\rangle$ is 0 with probability $(n-k)/n$ and otherwise, it is uniform on F. □

Remark 2. Notice that if $G=(f;\sigma)$ with $\sigma=\tau_n\circ\dots\circ\tau_2$ (cf. (2)), then X_j is the indicator function of $\tau_{n-j+1}=\mathrm{Id}$.

Unitary and orthogonal groups

Take $\mathcal{G}=U(n,\mathbb{C})$. Then $\mu_{\mathcal{H}_k}=f_k(\mu_{U(n-k,\mathbb{C})})$ where $f_k : A\in U(n-k,\mathbb{C})\mapsto \mathrm{Id}_k\oplus A$. As all reflections with respect to a hyperplane of $\mathbb{C}^{n-k}$ are elements of $U(n-k,\mathbb{C})$, one can apply Theorem 1 and Lemma 2. The Hermitian products $\langle e_k, h_k(e_k)\rangle$ are distributed as the first coordinate of the first vector of an element of $U(n-k,\mathbb{C})$, that is to say the first coordinate of the $(n-k)$-dimensional unit complex sphere with uniform measure :

$$\langle e_{k+1}, H_k(e_{k+1})\rangle \overset{\text{law}}{=} \mathrm{e}^{\mathrm{i}\omega_n}\sqrt{B_{1,n-k-1}}$$

with ω_n uniform on $(-\pi,\pi)$ and independent of $B_{1,n-k-1}$, a beta variable with parameters 1 and $n-k-1$.

Therefore, as a consequence of Theorem 2, we obtain the following decomposition formula derived in [10]. For $g\in U(n,\mathbb{C})$ which is $\mu_{U(n,\mathbb{C})}$ distributed, one has

$$\det(\mathrm{Id}_n - G) \overset{\text{law}}{=} \prod_{k=1}^n \left(1-\mathrm{e}^{\mathrm{i}\omega_k}\sqrt{B_{1,k-1}}\right),$$

with $\omega_1,\dots,\omega_n,B_{1,0},\dots,B_{1,n-1}$ independent random variables, the ω_k's uniformly distributed on $(-\pi,\pi)$ and the $B_{1,j}$'s $(0\le j\le n-1)$ being beta distributed with parameters 1 and j (by convention, $B_{1,0} = 1$).

A similar reasoning may be applied to $SO(2n)$ (with the complex unit spheres replaced by the real ones) to yield the following: let $G \in SO(2n)$ be $\mu_{SO(2n)}$ distributed, then (Corollary 6.2 in [10])

$$\det(\mathrm{Id}_{2n} - G) \stackrel{\text{law}}{=} 2 \prod_{k=2}^{2n} \left(1 - \epsilon_k \sqrt{B_{\frac{1}{2}, \frac{k-1}{2}}}\right) .$$

The quaternionic group

Our goal with this example is to solve Question 1 which was raised in the Introduction. To this end we establish an analogous to Lemma 2 and use the fact that $U(n, \mathbb{H}) \cong \mathrm{USp}(2n)$ which is also denoted $Sp(n)$, see for instance [25] Theorem 2. Then we apply Theorem 1. Let us give details. Recall that the symplectic group $\mathrm{USp}(2n, \mathbb{C})$ is defined as $\mathrm{USp}(2n, \mathbb{C}) = \{U \in U(2n, \mathbb{C}) \mid U J_n {}^{\mathrm{t}}U = J_n\}$, with

$$J_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} . \tag{7}$$

Let

$$\phi : \begin{cases} \mathbb{H} & \to \quad M(2, \mathbb{C}) \\ a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d & \mapsto \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix} \end{cases} ,$$

be the usual representation of quaternions. It is a continuous injective ring morphism such that $\phi(\bar{x}) = \phi(x)^*$. It induces the ring morphism

$$\Phi : \begin{cases} M(n, \mathbb{H}) & \to \quad M(2n, \mathbb{C}) \\ (a_{ij})_{1 \le i,j \le n} & \mapsto (\phi(a_{ij}))_{1 \le i,j \le n} \end{cases} .$$

In particular

$$\Phi(U(n, \mathbb{H})) = \{G \in U(2n, \mathbb{C}) : G \tilde{Z}_n {}^{\mathrm{t}}G = \tilde{Z}_n\}$$

where $\tilde{Z}_n = J_1 \oplus \cdots \oplus J_1$ and $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\tilde{Z}_n$ is conjugate to J_n, defined by (7), the set $\Phi(U(n, \mathbb{H}))$ is therefore conjugate to $\mathrm{USp}(2n, \mathbb{C})$. We can therefore consider $\det(I - \Phi(G))$

Lemma 3. *If for $k = 1, \dots, n-1$, $R_k \in \mathcal{R}_k$, then*

$$\det(\mathrm{Id}_{2n} - \Phi(R_0 \cdots R_{n-1})) = \prod_{k=0}^{n-1} \det(\mathrm{Id}_2 - \phi(\langle e_{k+1}, R_k(e_{k+1}) \rangle)) . \tag{8}$$

Proof. Let us first remark that the canonical basis $e_1, \dots, e_n$ of $\mathbb{H}^n$ is mapped by Φ into the canonical basis $\varepsilon_1, \dots, \varepsilon_{2n}$ of $\mathbb{C}^{2n}$, where the $2n \times 2$ matrix $[\varepsilon_{2k-1}, \varepsilon_{2k}]$ is exactly $\Phi(e_k)$. Moreover, if R is a proper reflection (leaving invariant an hyperplane), $\Phi(R)$ is a bireflection of $\mathbb{C}^{2n}$ i.e. a unitary transformation leaving invariant a vector space of codimension 2.

We start with

$$\det\big((\mathrm{Id}_{2n} - \Phi(RH)\big) = \det\big(\mathrm{Id}_{2n} - \Phi(R)\Phi(H)\big) = \det \Phi(H) \det\big(\Phi(H^*) - \Phi(R)\big)$$

Since H (hence H^*) stabilizes e_1, then $\Phi(H)$ (and $\Phi(H)^*$) stabilizes ε_1 and ε_2, so we have:

i) $(H^* - R)(e_1) = e_1 - R(e_1) =: a = [a_1, a_2]$ (say), hence, for $i = 1, 2$,

$$(\Phi(H^*) - \Phi(R))(\varepsilon_i) = \varepsilon_i - \Phi(R)(\varepsilon_i) =: a_i \,.$$

ii) Assume that $\langle e_1, w\rangle = 0$. Trivially, $\langle e_1, H^*(w)\rangle = 0$ hence $\Phi(H^*)(w)$ is a matrix whose column vectors are orthogonal to ε_1 and ε_2. Moreover, since R is a quaternionic reflection, $R(w) - w$ is a (right) scalar multiple of a (see [14] Proposition 1.6), so $\Phi\,(R(w) - w)$ is a $2n \times 2$ matrix whose columns are in Span (a_1, a_2).

By the multilinearity of the determinant, we get

$$\det\big(\Phi(H)^* - \Phi(R)\big) = \det\left(\langle \epsilon_i, a_j\rangle_{1\le i,j\le 2}\right) \det(\pi(H^*) - \mathrm{Id}_{2n-2})$$

which yields

$$\det(\mathrm{Id}_n - \Phi(RH)) = \det(\mathrm{Id}_2 - \phi(\langle e_1, R(e_1)\rangle)) \det(\mathrm{Id}_{2n-2} - \pi(H)) \,.$$

Iterating, we can conclude. $\square$

Corollary 2. Symplectic group. *Let $G \in \mathrm{USp}(2n, \mathbb{C})$ be $\mu_{\mathrm{USp}(2n,\mathbb{C})}$ distributed. Then*

$$\det(\mathrm{Id}_{2n} - G) \overset{\text{law}}{=} \prod_{k=1}^{n} \left((a_k - 1)^2 + b_k^2 + c_k^2 + d_k^2\right),$$

where the vectors (a_k, b_k, c_k, d_k), $1 \le k \le n$ are independent and (a_k, b_k, c_k, d_k) are 4 coordinates of the 4k-dimensional real unit sphere endowed with the uniform measure.

Remark 3. We have $(a_k, b_k, c_k, d_k) \overset{\text{law}}{=} \frac{1}{\sqrt{\mathcal{N}_1^2 + \cdots + \mathcal{N}_{4k}^2}} (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4)$, with the $\mathcal{N}_i's$ i.i.d. $\mathcal{N}(0, 1)$. Now, since for $p < q$

$$\frac{\mathcal{N}_1^2 + \cdots + \mathcal{N}_p^2}{\mathcal{N}_1^2 + \cdots + \mathcal{N}_q^2} \overset{\text{law}}{=} B_{\frac{p}{2}, \frac{q-p}{2}} \,,$$

we get the somehow more tractable identity in law

$$\det(\mathrm{Id}_{2n}-G)\overset{\text{law}}{=}\prod_{k=1}^{n}\left(\left(1+\epsilon_k\sqrt{B_{\frac12,2k-\frac12}}\right)^2+\left(1-B_{\frac12,2k-\frac12}\right)B'_{\frac32,2k-2}\right),$$

with all variables independent, $\mathbb{P}(\epsilon_k=1)=\mathbb{P}(\epsilon_k=-1)=1/2$.

This method can be applied to other interesting groups such as $\mathrm{USp}(2n,\mathbb{R})=\{u\in U(2n,\mathbb{R})\mid uz\,{}^{\mathrm{t}}u=z\}$ thanks to the morphism

$$\phi:\begin{cases}\quad\mathbb{C}\quad\;\to M(2,\mathbb{R})\\ a+\mathrm{i}b\mapsto\begin{pmatrix}a&-b\\ b&a\end{pmatrix}\end{cases}.$$

The traditional representation of the quaternions in $M(4,\mathbb{R})$

$$\phi:\begin{cases}\qquad\quad\mathbb{C}\qquad\qquad\to\qquad M(4,\mathbb{R})\\ a+\mathrm{i}b+\mathrm{j}c+\mathrm{k}d\mapsto\begin{pmatrix}a&-b&-c&-d\\ b&a&-d&-c\\ c&d&a&-b\\ d&-c&b&a\end{pmatrix}\end{cases}$$

gives another identity in law for a compact subgroup of $U(4n,\mathbb{R})$.

2.5 The generalized Ewens measure

In this section we wish to define a generalization of the Ewens measure on $U(n,K)$ and some of its subgroups which will agree with the classical definition on the symmetric group. We first recall the definition of the Ewens measure on the symmetric group and how it can be generated.

The Ewens measure on $\mathcal{S}_n$

Recall (see (2) Section 1) that every permutation $\sigma\in\mathcal{S}_n$ can be decomposed in the following way:

$$\sigma=\tau_n\circ\cdots\circ\tau_2 \tag{9}$$

where for $k=2,\dots,n$, τ_k is either the identity or the transposition (k,m_k) for some $m_k\in[k-1]$. In the first case we will say by extension that it is the transposition (k,m_k) with $m_k=k$. The number of cycles in the decomposition of σ is denoted k_σ. The system of Ewens measures of parameter $\theta>0$ consists in choosing the $m_k, k=1,\dots,n$ independently, with distribution

$$\mathbb{P}(m_k=k)=\frac{\theta}{\theta+k-1}\ ;\ \mathbb{P}(m_k=j)=\frac{1}{\theta+k-1},j=1,\dots,k-1.$$

It is known that the induced probability on $\mathcal{S}_n$ is

$$\mu_n^\theta(\sigma)=\frac{\theta^{k_\sigma}}{(\theta)_n}. \tag{10}$$

The generalized Ewens measure

In the following, $\mathcal{G}$ is any subgroup of $U(n,K)$. Take $\delta \in \mathbb{C}$ such that

$$0 < \mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}_n - G)^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{G})^{\delta}\right) < \infty. \tag{11}$$

For $0 \leq k \leq n-1$ we note

$$\exp_\delta^{(k)} : \begin{cases} \mathcal{G} \to \mathbb{R}^+ \\ G \mapsto (1 - \langle e_{k+1}, G(e_{k+1})\rangle)^{\overline{\delta}}(1 - \overline{\langle e_{k+1}, G(e_{k+1})\rangle})^{\delta} \end{cases}.$$

Moreover, define $\det_\delta$ as the function

$$\det{}_\delta : \begin{cases} \mathcal{G} \to \mathbb{R}^+ \\ G \mapsto \det(\mathrm{Id}_n - G)^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{G})^{\delta} \end{cases}.$$

Then the following generalization of Theorem 1 (which corresponds to the case $\delta = 0$) holds. However, note that, contrary to Theorem 1, in the following result we need that the coherent measures be supported by the set of reflections.

Theorem 3. Generalized Ewens sampling formula. *Let $\mathcal{G}$ be a subgroup of $U(n,K)$ checking condition (R) and (11). Let $(\nu_0,\dots,\nu_{n-1})$ be a sequence of measures coherent with $\mu_{\mathcal{G}}$, with $\nu_k(\mathcal{R}_k) = 1$. We note $\mu_{\mathcal{G}}^{(\delta)}$ the $\det_\delta$-sampling of $\mu_{\mathcal{G}}$ and $\nu_k^{(\delta)}$ the $\exp_\delta^{(k)}$-sampling of ν_k. Then*

$$\nu_0^{(\delta)} \star \nu_1^{(\delta)} \star \dots \star \nu_{n-1}^{(\delta)} = \mu_{\mathcal{G}}^{(\delta)},$$

i.e., for all test functions f on $\mathcal{G}$,

$$\mathbb{E}_{\nu_0^{(\delta)}\star\dots\star\nu_{n-1}^{(\delta)}}\left(f(R_0R_1\dots R_{n-1})\right) = \frac{\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(G)\det(\mathrm{Id}_n - G)^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{G})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}_n - G)^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{G})^{\delta}\right)}.$$

Proof. From Theorem 1, $G \overset{\text{law}}{=} R_0 \dots R_{n-1}$, hence

$$\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(G)\det(\mathrm{Id}_n - G)^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{G})^{\delta}\right) =$$
$$\mathbb{E}_{\nu_0\star\dots\star\nu_{n-1}}\left(f(R_0\dots R_{n-1})\det(\mathrm{Id}_n - R_0\dots R_{n-1})^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{R_0\dots R_{n-1}})^{\delta}\right).$$

From Lemma 2, $\det(\mathrm{Id}_n - R_0\dots R_{n-1}) = \prod_{k=0}^{n-1}(1 - \langle e_{k+1}, R_k(e_{k+1})\rangle)$, hence

$$\mathbb{E}_{\nu_0\star\dots\star\nu_{n-1}}\left(f(R_0\dots R_{n-1})\det(\mathrm{Id}_n - R_0\dots R_{n-1})^{\overline{\delta}}\det(\mathrm{Id}_n - \overline{R_0\dots R_{n-1}})^{\delta}\right) =$$
$$\mathbb{E}_{\nu_0\star\dots\star\nu_{n-1}}\left(f(R_0\dots R_{n-1})\prod_{k=0}^{n-1}\exp_\delta^{(k)}(R_k)\right).$$

By the definition of the measures $\nu_k^{(\delta)}$, this is the desired result. $\square$

Before exploring properties of this measure, let us give two examples of δ-samplings.

First we check that we can recover the classical Ewens measure on the symmetric group. Consider $\mathcal{G} = \mathbb{Z}_2 \wr \mathcal{S}_n$. For $\delta > 0$, the δ-sampling in $\mathbb{Z}_2 \wr \mathcal{S}_n$ induces a $\theta = 2^{2\delta-1}$ sampling on $\mathcal{S}_n$.

Proposition 2. *For $\delta > 0$, the pushforward of $\mu^{(\delta)}_{\mathbb{Z}_2 \wr \mathcal{S}_n}$ by the projection $(f,\sigma) \mapsto \sigma$ is μ_n^θ with $\theta = 2^{2\delta-1}$.*

Similarly, if we associate with each transposition of the decomposition (9) a Rademacher variable, we get easily a sequence of reflections, and if ν_k denotes the k-th corresponding measure, then the system $(\nu_0, \cdots, \nu_{n-1})$ is coherent with $\mu_{\mathbb{Z}_2 \wr \mathcal{S}_n}$. The pushforward of $\nu_k^{(\delta)}$ under the projection is a transposition biased by θ, so we recover the Ewens sampling formula.

Proof. Recall that the generic element of $\mathbb{Z}_2 \wr \mathcal{S}_n$ is denoted (f,σ). Let $\mathcal{C}(\sigma)$ the set of cycles of σ. If $c = (d_1, \dots, d_j)$ is such a cycle, let $\ell(c) = j$ and $w(f;c) = \prod_1^j f(d_j)$. Then it is clear that

$$\det\left(x\mathrm{Id}_n - (f;\sigma)\right) = \prod_{c \in \mathcal{C}(\sigma)} \left(x^{\ell(c)} - w(f;c)\right) ,$$

and in particular,

$$\det\left(\mathrm{Id}_n - (f;\sigma)\right) = \begin{cases} 0 & \text{if} \quad \exists c \in \mathcal{C}(\sigma) : w(f;c) = 1 \\ 2^{k_\sigma} & \text{if } \forall c \in \mathcal{C}(\sigma) : w(f;c) = -1 \,. \end{cases} \tag{12}$$

Let $\mathbb{P}$ stand for $\mu_{\mathbb{Z}_2 \wr \mathcal{S}_n}$ i.e. the uniform distribution on $\mathbb{Z}_2 \wr \mathcal{S}_n$. For any test function F

$$\mathbb{E}\left(F(\sigma) |\det(\mathrm{Id}_n - (f,\sigma))|^{2\delta}\right) = \mathbb{E}\left[F(\sigma)\mathbb{E}\left(|\det(\mathrm{Id}_n - (f,\sigma))|^{2\delta} | \sigma\right)\right] .$$

Now, conditionally on σ, the weights of the cycles are independent Rademacher variables (i.e. ± 1 with probability $1/2$). So,

$$\mathbb{P}\left(\cap_{c \in \mathcal{C}(\sigma)} \{w(f,\sigma) = -1\} | \sigma\right) = 2^{-k_\sigma}$$

and, due to (12)

$$\mathbb{E}\left(|\det(\mathrm{Id}_n - (f,\sigma))|^{2\delta} | \sigma\right) = 2^{(2\delta-1)k_\sigma} ,$$

which easily yields

$$\mathbb{E}_{\mu^{(\delta)}_{\mathbb{Z}_2 \wr \mathcal{S}_n}} F(\sigma) = \int_{\mathcal{S}_n} F(\sigma) d\mu_n^\theta(\sigma) .$$

□

The fundamental example remains $U(n,\mathbb{C})$. In the following section, we will study the determinantal sructure of this model for $\mathfrak{Re}\,\delta > -1/2$. In [12] a precise analysis of the reflections involved in the decomposition is given. The case $\delta = 1$ has a specific interest. If $(\theta_1, \dots, \theta_n)$ are the eigenangles of a unitary matrix, we have

$$|\det(\mathrm{Id} - U)|^2 = \prod_{j=1}^{n} |1 - \mathrm{e}^{\mathrm{i}\theta_j}|^2 ,$$

which, thanks to the density of the eigenangles, yields

$$\begin{aligned}&\mathbb{E}_{\mu^{(1)}_{U(n)}} (f(\theta_1, \dots, \theta_n)) \\ &\qquad = \mathrm{cst} \int_{(-\pi,\pi)^n} f(\theta_1, \dots, \theta_n) \prod_{j<k} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^2 \prod_{l=1}^{n} |1 - e^{\mathrm{i}\theta_l}|^2 \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n .\end{aligned}$$

This means that the distribution of the eigenangles $(\theta_1, \dots, \theta_n)$ of a random matrix drawn according to $\mu^{(1)}_{U(n)}$ is the same as the distribution of the n first eigenangles $(\theta_1, \cdots, \theta_n)$ of a random matrix drawn according to $\mu_{U(n+1,\mathbb{C})}$, conditionally on $\theta_{n+1} = 0$, or, as seen in [38], as the distribution of $(\theta_1 - \theta_{n+1}, \cdots, \theta_n - \theta_{n+1})$. More generally, in [11], Bourgade gives a geometrical characterisation of this kind of measures for $\delta/2 \in \mathbb{N}$, defining the notion of conditional Haar measure.

Remark 4. A generalized Ewens sampling formula could also be stated for $\Phi(\mathcal{G})$, with $\mathcal{G}$ checking condition (R) and Φ the ring morphism previously defined.

3 A hypergeometric kernel

In this section, we study the correlations of the point process of eigenvalues under the measure $\mu^{(\delta)}_{U(n,\mathbb{C})}$ and answer Question 3 (see Introduction) asked by Borodin-Olshanski in [7] section 8. Let us recall some basic facts on determinantal processes and correlations, referring to the books [1] 4.2 or [6] or [16] chap. 4.

Let $\Lambda = \mathbb{R}$ or $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{\mathrm{e}^{\mathrm{i}\theta}; \theta \in [-\pi, \pi]\}$ and let us fix an integer n. The collection of eigenvalues $(\lambda_1, \dots, \lambda_n)$ of a random $n \times n$ Hermitian (resp. unitary) matrix can be viewed as a point process on Λ , i.e. a random counting measure $\nu_n = \delta_{\lambda_1} + \cdots + \delta_{\lambda_n}$. Let us consider a simple point process ν on Λ. If there exists a sequence of locally integrable functions ρ_k such that for any mutually disjoint family of subsets $D_1, \dots, D_k$ of Λ

$$\mathbb{E}\left[\prod_{i=1}^{k} \nu(D_i)\right] = \int_{\prod_{i=1}^{k} D_i} \rho_k(x_1, \dots, x_k) \mathrm{d}x_1 \dots \mathrm{d}x_k$$

then the functions ρ_k are called the correlation functions, or joint intensities of the point process. In this case, the process is said to be determinantal with kernel K if its correlation functions ρ_k are given by

$$\rho_k(x_1, \ldots, x_k) = \det_{i,j=1}^{k} K(x_i, x_j)\,.$$

For $\nu = \nu_n$ we denote the correlations by $\rho_{k,n}$ for $k \leq n$. When the joint density of the eigenvalues is proportional to

$$\prod_{k=1}^{n} w(x_k) \prod_{1 \leq j < k \leq n} |x_k - x_j|^2$$

for some weight w, the orthogonal poynomial method shows that the point process of eigenvalues is determinantal. The use of Cayley transform allows to connect Hermitian matrices and unitary matrices. We give a detailed description of the consequence of this connection for the corresponding eigenvalue processes in Subsection 3.1, and its impact on the circular Jacobi ensemble in Subsection 3.2. Finally, we study the asymptotic behavior in Subsection 3.3.

3.1 Determinantal processes and Cayley transform

We follow the approach of Forrester ([16] 2.5 and 4.1.4). We start with a weight (positive integrable function) $w^{\mathbb{T}}$ on $\mathbb{T}$. The pushforward of the measure

$$\prod_{j=1} w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta_j}) \prod_{1 \leq j < k \leq n} |\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^2 \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n$$

by the stereographic projection (Cayley transform)

$$\lambda = \mathrm{i}\frac{1 - \mathrm{e}^{\mathrm{i}\theta}}{1 + \mathrm{e}^{\mathrm{i}\theta}} = \tan\frac{\theta}{2}\ ;\ \mathrm{e}^{\mathrm{i}\theta} = \frac{1 + \mathrm{i}\lambda}{1 - \mathrm{i}\lambda}$$

gives the measure

$$2^{n^2} \prod_{j=1}^{n} w^{\mathbb{T}}\left(\frac{1 + \mathrm{i}\lambda_j}{1 - \mathrm{i}\lambda_j}\right)(1 + \lambda_j^2)^{-n} \prod_{1 \leq j < k \leq n} |\lambda_k - \lambda_j|^2 \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n.$$

We define the weight $w^{\mathbb{R}}$ on $\mathbb{R}$ as

$$w^{\mathbb{R}}(x) = (1 + x^2)^{-n} w^{\mathbb{T}}\left(\frac{1 + \mathrm{i}x}{1 - \mathrm{i}x}\right)\,.$$

Conversely

$$w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta}) = \left(\cos\frac{\theta}{2}\right)^{2n} w^{\mathbb{R}}\left(\tan\frac{\theta}{2}\right)\,.$$

If the monomials $1, x, \dots, x^n$ are in $L^2(w^{\mathbb{R}}(x)\mathrm{d}x)$, then the orthogonal polynomial method gives

$$\frac{1}{\mathcal{Z}_n^{\mathbb{R}}}\prod_{j=1}^n w^{\mathbb{R}}(\lambda_j)\prod_{1\le j<k\le n}|\lambda_k-\lambda_j|^2=\det\left(\widetilde{K}_n^{\mathbb{R}}(\lambda_j,\lambda_k)\right)_{1\le j,k\le n}$$

where $\mathcal{Z}_n^{\mathbb{R}}$ is a normalization constant and where

$$\begin{aligned}\widetilde{K}_n^{\mathbb{R}}(x,y)&=\sqrt{w^{\mathbb{R}}(x)w^{\mathbb{R}}(y)}\ K_n^{\mathbb{R}}(x,y)\\ K_n^{\mathbb{R}}(x,y)&=\sum_{\ell=0}^{n-1}p_\ell^{\mathbb{R}}(x)p_\ell^{\mathbb{R}}(y)\end{aligned}$$

and the $p_\ell^{\mathbb{R}}$ are orthonormal with respect to the measure $w^{\mathbb{R}}(x)\mathrm{d}x$. The Christoffel-Darboux formula gives another expression for the kernel

$$K_n^{\mathbb{R}}(x,y)=\frac{\kappa_{n-1}}{\kappa_n}\ \frac{p_n^{\mathbb{R}}(x)p_{n-1}^{\mathbb{R}}(y)-p_{n-1}^{\mathbb{R}}(x)p_n^{\mathbb{R}}(y)}{x-y}$$

where κ_j is the coefficient of x^j in $p_j^{\mathbb{R}}(x)$. In terms of the monic orthogonal polynomials $P_0,\cdots,P_n$, this yields

$$K_n^{\mathbb{R}}(x,y)=\sum_{\ell=0}^{n-1}\frac{P_\ell(x)P_\ell(y)}{\|P_\ell\|^2} \tag{13}$$

$$=\frac{P_n^{\mathbb{R}}(x)P_{n-1}^{\mathbb{R}}(y)-P_{n-1}^{\mathbb{R}}(x)P_n^{\mathbb{R}}(y)}{\|P_{n-1}\|^2(x-y)}\,. \tag{14}$$

Besides, on the unit circle, we consider the polynomials φ_ℓ (resp. Φ_ℓ) orthonormal (resp. monic orthogonal) with respect to the measure $w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta$, and their reciprocal defined by

$$\Phi_\ell^\star(z)=z^\ell\,\overline{\Phi_\ell(1/\bar z)}\ ,\ \varphi_\ell^\star(z)=z^\ell\,\overline{\varphi_\ell(1/\bar z)}\,.$$

We have then

$$\frac{1}{\mathcal{Z}_n^{\mathbb{T}}}\prod_{j=1}^n w^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta_j})\prod_{1\le j<k\le n}|\mathrm{e}^{\mathrm{i}\theta_k}-\mathrm{e}^{\mathrm{i}\theta_j}|^2=\det\left(\widetilde{K}_n^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta_j},\mathrm{e}^{\mathrm{i}\theta_k})\right)_{1\le j,k\le n}$$

with

$$\widetilde{K}_n^{\mathbb{T}}(z,\zeta)=\sqrt{w^{\mathbb{T}}(z)w^{\mathbb{T}}(\zeta)}\ K_n^{\mathbb{T}}(z,\zeta)$$

and

$$K_n^{\mathbb{T}}(z,\zeta)=\sum_{\ell=0}^{n-1}\overline{\varphi_\ell(z)}\varphi_\ell(\zeta)\,.$$

The Christoffel-Darboux formula is now

$$K_n^{\mathbb{T}}(z,\zeta) = \frac{\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) - \overline{\varphi_n(z)}\varphi_n(\zeta)}{1-\bar{z}\zeta} \tag{15}$$

(see [35] 1.12 and 3.2), or

$$K_n^{\mathbb{T}}(z,\zeta) = \frac{\overline{\Phi_n^*(z)}\Phi_n^*(\zeta) - \overline{\Phi_n(z)}\Phi_n(\zeta)}{\|\Phi_n\|^2(1-\bar{z}\zeta)}\,. \tag{16}$$

The kernel $\widetilde{K}_n^{\mathbb{R}}$ (resp. $\widetilde{K}_n^{\mathbb{T}}$) rules the correlation function $\rho_{n,m}^{\mathbb{R}}(\lambda_1,\cdots,\lambda_m)$ (resp. $\rho_{n,m}^{\mathbb{C}}(\mathrm{e}^{\mathrm{i}\theta_1},\cdots,\mathrm{e}^{\mathrm{i}\theta_m})$) for $m=1,\cdots,n$.

3.2 Our weights and their characteristics

For the sake of simplicity we use the polygamma symbol

$$\Gamma\begin{bmatrix} a,b,\cdots \\ c,d,\cdots \end{bmatrix} := \frac{\Gamma(a)\Gamma(b)\cdots}{\Gamma(c)\Gamma(d)\cdots}.$$

For $\delta = a+\mathrm{i}b \in \mathbb{C}$ with $a>-1/2$, we will consider two weights on $(-\pi,\pi)$

$$w_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta}) = (1-\mathrm{e}^{\mathrm{i}\theta})^{\overline{\delta}}(1-\mathrm{e}^{-\mathrm{i}\theta})^{\delta} = (2-2\cos\theta)^a \mathrm{e}^{-b(\pi\,\mathrm{sgn}\,\theta-\theta)} \tag{17}$$

$$w_2^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta}) = (1+\mathrm{e}^{\mathrm{i}\theta})^{\overline{\delta}}(1+\mathrm{e}^{-\mathrm{i}\theta})^{\delta} = (2+2\cos\theta)^a \mathrm{e}^{-b\theta} \tag{18}$$

These are "pure" Fisher-Hartwig functions. We can go from $w_1^{\mathbb{T}}$ to $w_2^{\mathbb{T}}$ by the transform

$$\theta \mapsto \tau := -\theta + \pi(\mathrm{sgn}\,\theta) \tag{19}$$

which carries the discontinuity in $\theta=0$ to the edges $\pm\pi$, so that

$$\mathrm{e}^{\mathrm{i}\theta} = -\mathrm{e}^{-\mathrm{i}\tau} \text{ and } w_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta}) = w_2^{\mathbb{T}}(\mathrm{e}^{-\mathrm{i}\tau})\,. \tag{20}$$

For $a>-1/2$, the Fourier coefficients of w_1 are known ([9] Lemma 2.1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta = (-1)^n \Gamma\begin{bmatrix} 1+\delta+\bar{\delta} \\ \bar{\delta}-n+1, \delta+n+1 \end{bmatrix}.$$

With

$$c(\delta) = \frac{1}{2\pi}\Gamma\begin{bmatrix} 1+\delta, 1+\bar{\delta} \\ 1+\delta+\bar{\delta} \end{bmatrix},$$

the function $\widetilde{w}_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta}) = c(\delta)w_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})$ is a probability density on $(-\pi,\pi)$. For w_2, we note that

$$\int_{-\pi}^{\pi} w_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta = (-1)^n \int_{-\pi}^{\pi} w_2^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{\mathrm{i}n\tau}\mathrm{d}\tau.$$

Moreover we go from one system of polynomials to the other by the mapping $z \mapsto -z$.

It is known from [4] p. 304 and [5] p.31-34 that for $n \geq 0$ the n-th orthonormal polynomial with respect to $\widetilde{w}_1^{\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta$ is

$$\Phi_n(z) = \Gamma \begin{bmatrix} \delta + n, \bar{\delta} + 1 \\ \bar{\delta} + n + 1, \delta \end{bmatrix} \ {}_2F_1 \left(\begin{matrix} -n, \bar{\delta} + 1 \\ 1 - n - \delta \end{matrix} ; \ z \right) \tag{21}$$

with

$$\|\Phi_n\|^2 = \Gamma \begin{bmatrix} \delta + \bar{\delta} + n + 1, n + 1, \bar{\delta} + 1, \delta + 1 \\ \bar{\delta} + n + 1, \delta + n + 1, \delta + \bar{\delta} + 1 \end{bmatrix}, \tag{22}$$

(see also [16] Prop. 4.8 in the case δ real). With the complement formula (48) we get the other form

$$\Phi_n(z) = \Gamma \begin{bmatrix} \delta + \bar{\delta} + 1 + n, \bar{\delta} + 1 \\ \bar{\delta} + n + 1, \delta + \bar{\delta} + 1 \end{bmatrix} \ {}_2F_1 \left(\begin{matrix} -n, \bar{\delta} + 1 \\ \delta + \bar{\delta} + 1 \end{matrix} ; \ 1 - z \right). \tag{23}$$

In view of (47) and (21) we identify Φ_n^* as

$$\Phi_n^*(z) = \ {}_2F_1 \left(\begin{matrix} -n, \bar{\delta} \\ -n - \delta \end{matrix} ; \ z \right), \tag{24}$$

or, using (48) again

$$\Phi_n^*(z) = \Gamma \begin{bmatrix} \delta + \bar{\delta} + 1 + n, \delta + 1 \\ \delta + n + 1, \delta + \bar{\delta} + 1 \end{bmatrix} \ {}_2F_1 \left(\begin{matrix} -n, \bar{\delta} \\ \delta + \bar{\delta} + 1 \end{matrix} ; \ 1 - z \right). \tag{25}$$

Borodin and Olshanski considered the following weight on $\mathbb{R}$:

$$2^{-\delta-\bar{\delta}} w_2^{\mathbb{R}}(x) = (1 + \mathrm{i}x)^{-\delta - n}(1 - \mathrm{i}x)^{-\bar{\delta} - n}. \tag{26}$$

Since this weight depends on n, the reference measure has only a finite set of moments so that there is only a finite set of orthogonal polynomials (these are the pseudo-Jacobi polynomials)

$$p_m(x) = (x - \mathrm{i})^m \ {}_2F_1 \left(\begin{matrix} -m, \delta + n - m \\ \delta + \bar{\delta} + 2n - 2m \end{matrix} ; \ \frac{2}{1 + \mathrm{i}x} \right) \tag{27}$$

$m < a + n - \frac{1}{2}$. Let us call $\widetilde{K}_{2,n}^{\mathbb{R}}$ the corresponding kernel.

3.3 Asymptotic behavior

For the weight $w_2^{\mathbb{R}}$, Borodin and Olshanski considered the (thermodynamic) scaling limit $\lambda \mapsto n\lambda$ and proved ([7] Theorem 2.1)

Theorem 4 (Borodin-Olshanski). *Let $\mathfrak{Re}\, \delta > -1/2$.*

1. We have

$$\lim_n \ (\operatorname{sgn}\, x \ \operatorname{sgn}\, y)^n n \widetilde{K}_{2,n}^{\mathbb{R}}(nx, ny) = \widetilde{K}_{\infty}^{\mathbb{R}}(x, y) \tag{28}$$

uniformly for x, y in compact sets of $\mathbb{R}^\star \times \mathbb{R}^\star$, where (for $x \neq y$)

$$\widetilde{K}_\infty^{\mathbb{R}}(x,y) := \frac{1}{2\pi} \Gamma \begin{bmatrix} \delta+1, \bar{\delta}+1 \\ \delta+\bar{\delta}+1, \delta+\bar{\delta}+2 \end{bmatrix} \frac{\widetilde{P}(x)Q(y) - Q(x)\widetilde{P}(y)}{x-y} \tag{29}$$

$$\widetilde{P}(x) = \left|\frac{2}{x}\right|^{\frac{\delta+\bar{\delta}}{2}} \mathrm{e}^{-\frac{\mathrm{i}}{x}+\pi\frac{(\delta-\bar{\delta})\operatorname{sgn} x}{4}} \, {}_1F_1\left(\begin{matrix}\delta \\ \delta+\bar{\delta}+1\end{matrix} ; \frac{2\mathrm{i}}{x}\right) \tag{30}$$

$$Q(x) = \frac{2}{x}\left|\frac{2}{x}\right|^{\frac{\delta+\bar{\delta}}{2}} \mathrm{e}^{-\frac{\mathrm{i}}{x}+\pi\frac{(\delta-\bar{\delta})\operatorname{sgn} x}{4}} \, {}_1F_1\left(\begin{matrix}\delta+1 \\ \delta+\bar{\delta}+2\end{matrix} ; \frac{2\mathrm{i}}{x}\right). \tag{31}$$

2. The limiting correlation is given by

$$\lim_n n^m \rho_{n,m}^{\mathbb{R}}(n\lambda_1, \cdots, n\lambda_m) = \det\left(\widetilde{K}_\infty^{\mathbb{R}}(\lambda_i, \lambda_j)\right)_{1 \le i,j \le m}. \tag{32}$$

The kernel $\widetilde{K}_\infty^{\mathbb{R}}(1/x, 1/y)$ is called the *confluent hypergeometric kernel* in [8].

For the circular model, we choose the set-up w_1 for the sake of consistency with the above sections. The singularity is in $z = 1$ i.e. $\theta = 0$. To study the asymptotic behavior of the point process on $\mathbb{T}$ at the singularity (edge) we have two ways: either take the thermodynamic scaling $\theta \mapsto \theta/n$, or use the result on $\mathbb{R}$.

Theorem 5. *Let $\mathfrak{Re}\,\delta > -1/2$.*

1. With the weight w_1,

$$\lim_n n^{-1} \widetilde{K}_n^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta/n}, \mathrm{e}^{\mathrm{i}\tau/n}) = \widetilde{K}_\infty^{\mathbb{T}}(\theta, \tau) \tag{33}$$

with, for $\theta \neq \tau$

$$\widetilde{K}_\infty^{\mathbb{T}}(\theta,\tau) = \frac{1}{2\mathrm{i}\pi} \Gamma \begin{bmatrix} 1+\delta, 1+\bar{\delta} \\ 1+\delta+\bar{\delta}, 1+\delta+\bar{\delta} \end{bmatrix} \frac{P^{\mathbb{T}}(\theta)\overline{P^{\mathbb{T}}(\tau)} - \overline{P^{\mathbb{T}}(\theta)}P^{\mathbb{T}}(\tau)}{\theta-\tau} \tag{34}$$

where

$$P^{\mathbb{T}}(\theta) := |\theta|^{\frac{\delta+\bar{\delta}}{2}} \mathrm{e}^{\mathrm{i}\frac{\theta}{2} - \frac{\pi}{4}(\delta-\bar{\delta})\operatorname{sgn}\theta} \, {}_1F_1\left(\begin{matrix}\delta \\ \delta+\bar{\delta}+1\end{matrix} ; -\mathrm{i}\theta\right) = \widetilde{P}\big(-2\theta^{-1}\big), \tag{35}$$

and

$$\widetilde{K}_\infty^{\mathbb{T}}(\theta,\theta) = \frac{|\theta|^{\delta+\bar{\delta}}}{2\pi} \Gamma \begin{bmatrix} 1+\delta, 1+\bar{\delta} \\ 1+\delta+\bar{\delta}, 1+\delta+\bar{\delta} \end{bmatrix} \mathfrak{Re}$$

$$\left[{}_1F_1\left(\begin{matrix}\delta \\ \delta+\bar{\delta}+1\end{matrix} ; -\mathrm{i}\theta\right)\left[{}_1F_1\left(\begin{matrix}\bar{\delta} \\ \delta+\bar{\delta}+1\end{matrix} ; \mathrm{i}\theta\right) - 2\, {}_1F_1\left(\begin{matrix}\bar{\delta}+1 \\ \delta+\bar{\delta}+2\end{matrix} ; \mathrm{i}\theta\right)\right]\right] \tag{36}$$

2. The limiting correlation is given by

$$\lim_n n^m \rho_{n,m}^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta_1/n}, \cdots, \mathrm{e}^{\mathrm{i}\theta_m/n}) = \det\left(\widetilde{K}_\infty^{\mathbb{T}}(\theta_i, \theta_j)\right)_{1\le i,j\le m} . \tag{37}$$

Proof. We begin with a direct proof of (33) when $\theta \neq \tau$, and then proceed with the proof of (33) when $\theta = \tau$, which directly yields (37) and we end with an alternate proof of (37) using (32) and the Cayley transform.

1) The following lemma describes the asymptotical behavior of the quantities entering in the kernel.

Lemma 4. *When $n \to \infty$*

$$\lim_n \|\Phi_n\|^2 = \Gamma\begin{bmatrix}\bar{\delta}+1, \delta+1\\ \delta+\bar{\delta}+1\end{bmatrix} \tag{38}$$

Moreover if $n\theta_n \to \theta$, then (uniformly for θ in a compact set)

$$\lim n^{-\delta}\Phi_n(\mathrm{e}^{\mathrm{i}\theta_n}) = \Gamma\begin{bmatrix}\bar{\delta}+1\\ \delta+\bar{\delta}+1\end{bmatrix} {}_1F_1\left(\begin{matrix}\bar{\delta}+1\\ \delta+\bar{\delta}+1\end{matrix} ;\ \mathrm{i}\theta\right), \tag{39}$$

$$\lim n^{-\bar{\delta}}\Phi_n^\star(\mathrm{e}^{\mathrm{i}\theta_n}) = \Gamma\begin{bmatrix}\delta+1\\ \delta+\bar{\delta}+1\end{bmatrix} {}_1F_1\left(\begin{matrix}\bar{\delta}\\ \delta+\bar{\delta}+1\end{matrix} ;\ \mathrm{i}\theta\right), \tag{40}$$

$$\lim n^{-\bar{\delta}+1}(\Phi_n^\star)'(\mathrm{e}^{\mathrm{i}\theta_n}) = \bar{\delta}\Gamma\begin{bmatrix}\delta+1\\ \delta+\bar{\delta}+2\end{bmatrix} {}_1F_1\left(\begin{matrix}\bar{\delta}+1\\ \delta+\bar{\delta}+2\end{matrix} ;\ \mathrm{i}\theta\right). \tag{41}$$

Proof. Let us first recall that, as $n \to \infty$,

$$\frac{\Gamma(c+n)}{\Gamma(n)} \sim n^c , \tag{42}$$

which gives immediately (38). The limits in (39) and (40) are then consequences of (23), (25) and the limiting relation (50). Besides, in view of (49) and (25),

$$(\Phi_n^\star)'(z) = \frac{n\bar{\delta}}{\delta+\bar{\delta}+1}\Gamma\begin{bmatrix}\delta+\bar{\delta}+n, \delta+1\\ \delta+\bar{\delta}+1, \delta+n+1\end{bmatrix} {}_2F_1\left(\begin{matrix}-n+1, \bar{\delta}+1\\ \delta+\bar{\delta}+2\end{matrix} ;\ 1-z\right) .$$

It remains to apply (50). □

A) For $\theta \neq \tau$, we have, by the Christoffel-Darboux formula (15):

$$\lim_n \mathrm{i}(\theta-\tau)\Gamma(\delta+\bar{\delta}+1)n^{-(\delta+\bar{\delta}+1)}K_n^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta/n}, \mathrm{e}^{\mathrm{i}\tau/n}) =$$
$${}_1F_1\left(\begin{matrix}\delta\\ \delta+\bar{\delta}+1\end{matrix} ;\ -\mathrm{i}\theta\right) {}_1F_1\left(\begin{matrix}\bar{\delta}\\ \delta+\bar{\delta}+1\end{matrix} ;\ \mathrm{i}\tau\right)$$
$$- {}_1F_1\left(\begin{matrix}\delta+1\\ \delta+\bar{\delta}+1\end{matrix} ;\ -\mathrm{i}\theta\right) {}_1F_1\left(\begin{matrix}\bar{\delta}+1\\ \delta+\bar{\delta}+1\end{matrix} ;\ \mathrm{i}\tau\right)$$

Now, applying the Kummer's formula (52)

$$
{}_1F_1\left(\begin{matrix}\delta+1\\ \delta+\bar\delta+1\end{matrix}\ ;\ -\mathrm{i}\theta\right)=\mathrm{e}^{-\mathrm{i}\theta}\ {}_1F_1\left(\begin{matrix}\bar\delta\\ \delta+\bar\delta+1\end{matrix}\ ;\ \mathrm{i}\theta\right)
$$
$$
{}_1F_1\left(\begin{matrix}\bar\delta+1\\ \delta+\bar\delta+1\end{matrix}\ ;\ \mathrm{i}\tau\right)=\mathrm{e}^{\mathrm{i}\tau}\ {}_1F_1\left(\begin{matrix}\delta\\ \delta+\bar\delta+1\end{matrix}\ ;\ -\mathrm{i}\tau\right)
$$

Besides we have (recall that we used $\widetilde{w}_1$)

$$
\frac{\widetilde{K}_n^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta/n},\mathrm{e}^{\mathrm{i}\tau/n})}{K_n^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta/n},\mathrm{e}^{\mathrm{i}\tau/n})}=c(\delta)\sqrt{w_1(\mathrm{e}^{\mathrm{i}\theta/n})w_1(\mathrm{e}^{\mathrm{i}\tau/n})}
$$

and from the very definition of w_1

$$
\lim n^{2(\delta+\bar\delta)}w_1(\mathrm{e}^{\mathrm{i}\theta/n})w_1(\mathrm{e}^{\mathrm{i}\tau/n})=|\theta\tau|^{2\mathfrak{Re}\delta}\mathrm{e}^{-\mathfrak{Im}\delta\pi(\operatorname{sgn}\theta+\operatorname{sgn}\tau)}
$$

We conclude that (33) holds true.

B) On the diagonal In the following z and ζ are elements of $\mathbb{T}$. If F and G are differentiable functions on $\mathbb{T}$, the de l'Hospital rule gives

$$
\lim_{\zeta\to z}\frac{F(z)G(\zeta)-F(\zeta)G(z)}{z-\zeta}=F'(z)G(z)-F(z)G'(z)\,.
$$

Taking

$$
F(z)=z^{-n}\Phi_n(z)\ ,\ G(z)=\overline{\Phi_n(z)}\,,
$$

so that

$$
F'(z)=-nz^{-n-1}\Phi_n(z)+z^{-n}\Phi_n'(z)\ ,\ G'(z)=-z^{-2}\overline{\Phi_n'(z)}
$$

we get the value of the kernel on the diagonal:

$$
\begin{aligned}
\lim_{\zeta\to z}\frac{\overline{\Phi_n^*(z)}\Phi_n^*(\zeta)-\overline{\Phi_n(z)}\Phi_n(\zeta)}{1-\bar z\zeta}&=-n|\Phi_n(z)|^2+2\mathfrak{Re}[\overline{\Phi_n(z)}z\Phi_n'(z)]\\
&=n|\Phi_n^*(z)|^2-2\mathfrak{Re}[\overline{\Phi_n^*(z)}z(\Phi_n^*)'(z)].
\end{aligned}\tag{43}
$$

It remains to apply the lemma.

Notice that

$$
\lim_n n^{-(1+\delta+\bar\delta)}K_n^{\mathbb{T},1}(1,1)=\frac{1}{\Gamma(\delta+\bar\delta+2)}\,.
$$

2) *Alternate proof of (37)*

The pushforward of the measure

$$
\rho_n^{\mathbb{R},2}(x_1,\cdots,x_n)dx_1\dots dx_n
$$

by the Cayley transform is,

$$2^{-n}\rho_n^{\mathbb{R},2}\left(\tan\frac{\theta_1}{2},\cdots,\tan\frac{\theta_n}{2}\right)\prod_{k=1}^{n}\cos^{-2}\frac{\theta_k}{2}\ \mathrm{d}\theta_1\ldots\mathrm{d}\theta_n$$

which, at the level of kernels gives

$$\rho_{n,m}^{\mathbb{T},2}(\mathrm{e}^{\mathrm{i}\theta_1},\cdots,\mathrm{e}^{\mathrm{i}\theta_m}))=\det\left[\widetilde{K}_n^{\mathbb{R},2}\left(\tan\frac{\theta_i}{2},\tan\frac{\theta_j}{2}\right)\frac{1}{2\cos\theta_i\cos\theta_j}\right]_{1\le i,j\le m}.$$

Coming back to the superscript 1 with the help of (19) we obtain

$$\rho_{n,m}^{\mathbb{T},1}(\mathrm{e}^{\mathrm{i}\theta_1},\cdots,\mathrm{e}^{\mathrm{i}\theta_m}))=\det\left[H_n(\theta_i),H_n(\theta_j)\right]_{1\le i,j\le m}$$

with

$$H_n(\theta,\theta')=\widetilde{K}_n^{\mathbb{R},2}\left(-\cot\frac{\theta}{2},-\cot\frac{\theta'}{2}\right)\frac{1}{2|\sin\frac{\theta}{2}\sin\frac{\theta'}{2}|}.$$

Let us rescale the angles. Since $\lim_n n\tan\frac{\theta}{n}=\theta$, $\lim_n n\tan\frac{\theta'}{n}=\theta'$ and since the limit in (28) is uniform on compact subsets, we get

$$\lim\frac{1}{n}H_n\left(\frac{\theta}{n},\frac{\theta'}{n}\right)=\frac{2}{|\theta\theta'|}\widetilde{K}_\infty^{\mathbb{R}}\left(-\frac{2}{\theta},-\frac{2}{\theta'}\right).$$

We remark that $P^{\mathbb{T}}(\theta)=\widetilde{P}(x)$ with $x\theta=-2$. Moreover, from (53), we have

$$\frac{\mathrm{i}}{\bar{\delta}+\delta+1}Q(x)=\overline{P^{\mathbb{T}}(\theta)}-P^{\mathbb{T}}(\theta)$$

so that, if $\tau=-2/y$

$$\frac{\mathrm{i}}{\bar{\delta}+\delta+1}\left[\widetilde{P}(x)Q(y)-\widetilde{P}(y)Q(x)\right]=P^{\mathbb{T}}(\theta)\overline{P^{\mathbb{T}}(\tau)}-P^{\mathbb{T}}(\tau)\overline{P^{\mathbb{T}}(\theta)}$$

and consequently

$$\frac{\theta\tau}{2}\widetilde{K}_\infty^{\mathbb{T}}(\theta,\tau)=\widetilde{K}_\infty^{\mathbb{R}}(x,y)\,. \tag{44}$$

□

Remark 5. 1. To have a graphical point of view of this kernel, we refer to [13] p.56–60.

2. In [26], the behavior of the limiting kernel on $\mathbb{R}$ is used to study asymptotics of the maximal eigenvalue of the generalized Cauchy ensemble.
3. An easy computation shows that for δ real, $\delta>-1/2$, we recover the Bessel kernel

$$K_\infty^{\mathbb{T}}=\frac{\pi}{2}\sqrt{\theta\tau}\frac{J_{\delta+\frac12}(\frac{\pi\theta}{2})J_{\delta-\frac12}(\frac{\pi\tau}{2})-J_{\delta-\frac12}(\frac{\pi\theta}{2})J_{\delta+\frac12}(\frac{\pi\tau}{2})}{2(\theta-\tau)},$$

and for $\delta=0$ the sine kernel

$$K_\infty^{\mathbb{T}}=\frac{\sin(\frac{\theta-\tau}{2})}{\pi(\theta-\tau)}.$$

4 Appendix: Hypergeometric functions

For a classical reference on hypergeometric functions, see [2].

The Gauss hypergeometric function is defined as

$$ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,;\,z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{45}$$

where $(x)_n$ stands for the Pochhammer symbol $(x)_k = x(x+1)\dots(x+k-1)$, with the convention $(x)_0 = 1$. When $a = -n \in -\mathbb{N}_0$, it is a polynomial

$$ {}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix}\,;\,z\right)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(b)_k}{(c)_k}z^k. \tag{46}$$

The following relations are useful:

$$ z^n\ {}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix}\,;\,z^{-1}\right)=(-1)^n\frac{(b)_n}{(c)_n}\ {}_2F_1\left(\begin{matrix}-n,-n-c+1\\ -n-b+1\end{matrix}\,;\,z\right) \tag{47}$$

$$ {}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix}\,;\,1-z\right)=\frac{(c-b)_n}{(c)_n}\ {}_2F_1\left(\begin{matrix}-n,b\\ -n+b+1-c\end{matrix}\,;\,z\right) \tag{48}$$

$$ \frac{d}{dz}\ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,;\,z\right)=\frac{ab}{c}\ {}_2F_1\left(\begin{matrix}a+1,b+1\\ c+1\end{matrix}\,;\,z\right). \tag{49}$$

It is known that, uniformly for z in a compact set, for b, c fixed

$$ \lim_N\ {}_2F_1\left(\begin{matrix}-N,b\\ c\end{matrix}\,;\,-\frac{z}{N}\right)=\ {}_1F_1\left(\begin{matrix}b\\ c\end{matrix}\,;\,z\right) \tag{50}$$

where

$$ {}_1F_1\left(\begin{matrix}b\\ c\end{matrix}\,;\,z\right)=\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k}\frac{z^k}{k!} \tag{51}$$

is the confluent hypergeometric function.

It satisfies Kummer's formula:

$$ \mathrm{e}^z\ {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\,;\,-z\right)=\ {}_1F_1\left(\begin{matrix}c-a\\ c\end{matrix}\,;\,z\right), \tag{52}$$

the recursion formula

$$ {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\,;\,z\right)=\ {}_1F_1\left(\begin{matrix}a-1\\ c\end{matrix}\,;\,z\right)+\frac{z}{c}\ {}_1F_1\left(\begin{matrix}a\\ c+1\end{matrix}\,;\,z\right), \tag{53}$$

and the derivative formula

$$ \frac{d}{dz}\ {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\,;\,z\right)=\frac{a}{c}\ {}_1F_1\left(\begin{matrix}a+1\\ c+1\end{matrix}\,;\,z\right). \tag{54}$$

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