Lecture 3

Fixed point:

A pt \( p \) is a fixed pt of the map \( f \) if

\[
 f(p) = p
\]

Graphically, the fixed pts are the intersections of \( y = f(x) \), \( y = x \).

Stability of Fixed Points

**Bifurcations:** \( p \) is a fixed pt of \( f \). If all pts sufficiently close to \( p \) are attracted to \( p \), then \( p \) is called an attractor (an attracting fixed pt or stable fixed pt) i.e.

\[
 \exists \varepsilon > 0, \text{ st. } \forall x \in N^\varepsilon(p) \left( \lim_{k \to \infty} f^k(x) = p \right)
\]

then \( p \) is a sink.

**NB:** \( N^\varepsilon(p) = \{ x \in \mathbb{R} \mid |x - p| < \varepsilon \} \), \( \varepsilon > 0 \).
- \( \varepsilon \) - Neighborhood.

**Def.** \( p \) - source (or repelling fixed pt or unstable fixed pt)

\[
 \forall \varepsilon > 0, \exists N^\varepsilon(p) \text{ st. } \forall x \in N^\varepsilon(p), x \neq p \text{ eventually maps outside } N^\varepsilon(p)
\textbf{NB:} \( f \) — a smooth map on \( \mathbb{R} \), \( f(p) = p \) fixed pt.

1. If \( |f'(p)| < 1 \), then \( p \) is a stable fixed pt.
2. If \( |f'(p)| > 1 \), then \( p \) is an unstable fixed pt.

\textbf{Pf.} Consider \( |f'(p)| < 1 \)

\[ \therefore f' \text{ is continuous} \quad \rightarrow \quad |f'(x)| < 1 \quad \text{for all } x \text{ near } p \]

\[ \text{i.e. } \exists \varepsilon > 0, \forall x \in N^\varepsilon(p), \text{ we have } \]
\[ |f'(x)| \leq c < 1 \]

\[ \forall x \in N^\varepsilon(p), x \neq p \quad f(x) = p \text{ fixed pt.} \]

\[ \frac{|f(x) - p|}{|x - p|} = \frac{|f(x) - f(p)|}{|x - p|} \]

\[ \text{The mean value theorem} \quad \frac{|f'(\xi)(x - p)|}{|x - p|} = |f'(\xi)| \leq c \]

\[ \exists \xi \in (x, p) \quad \xi \in N^\varepsilon(p) \]

\[ \therefore |f(x) - p| \leq c |x - p| \]

After \( k^{th} \) iterate, \( |f^k(x) - p| \leq c^k |x - p| \)

\[ \therefore 0 < c < 1 \quad c^k \rightarrow 0 \text{ as } k \rightarrow \infty \]

\[ \therefore p \text{ is stable.} \quad \Box \]
1° In the proof, we can see that the rate of convergence of $f^k(p)$ to $p$ is exponential as $k \to \infty$.

2° The theorem does not say what happens if $|f'(p)| = 1$.

\[ f_{n+1} = f(f_n), \quad f = \mathbb{R}, \quad f' \equiv 1 \]

Every point stays where it is.

\[ f(0) \]

\[ f(x) \]

\[ y = x \]

(\(p_0, f_0(p)\))

\[ y = x \]

(\(p_1, f_1(p)\))

\[ y = x \]

(\(p_2, f_2(p)\))

Neither stable nor unstable.
Example: Newton's Method.

To solve \( g(x) = 0 \) numerically.

To - initial guess

\[
\frac{g'(x_0)}{g(x_0)} = \frac{\Delta y}{\Delta x} = \frac{0 - g(x_0)}{x_1 - x_0}
\]

\[x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}\]

using \( x_1 \) with the same procedure to obtain \( x_2, x_3 \) etc.

\[x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}\]

\[\therefore \text{The map in Newton's method for finding root of } g(x) \text{ is}\]

\[f(x) = x - \frac{g(x)}{g'(x)}\]

\[\text{NB: } \text{the fixed pt } f(x_0) = x \text{ is the root of } g(x), g(x_0) = 0\]

\[\therefore f'(x) = \frac{g(x)g''(x)}{(g'(x))^2} \therefore f'(x) = 0\]

\[\therefore f(x) \text{ has a very rapidly converging fixed pt } x\]

\[\text{NB: } \text{a fixed pt } p \text{ of } f, f(p) = 0 \text{ is unnecessary.}\]
Periodic orbits

Example: \( f(x) = -x \) on \( \mathbb{R} \), \( x = 0 \) is a fixed pt

\[ f^2 = \text{an identity map} \]

\[ \forall x \neq 0 \text{ is a period } -2 \text{ pt.} \]

Definition: If \( f^k(p) = p \) and \( k \) is the smallest such positive integer, then the orbit with initial pt \( p \) is a period \( k \) orbit.

Note: The smallest positive integer \( k \) is also called "prime" period of \( p \). — Nothing to do with prime numbers.

Example: \( x_{n+1} = 1 - x^n \), find fixed pts, period-2 pts and period-3 pts?

\[ \therefore f(x) = 1 - x \]

Fixed pts: \( f(x) = x \) i.e. \( 1 - x^2 = x \)

\[ x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}. \]

Period-2 pts: \( f^2(x) = x \)
\[ f(x) = 1 - x^2, \quad f^2(x) = f(f(x)) = 1 - (1 - x^2)^2 = 2x^2 - x^4 \]

Solve \( f(x) = x \) i.e. \( 2x^2 - x^4 = x \)

\[ x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}, \quad x_3 = 0, \quad x_4 = 1. \]

\textbf{NB!} The fixed pts: \( f(p) = p \Rightarrow f^2(p) = f(p) = p \).

\[ x_3 = 0, \quad x_4 = 1 \text{ are period-2 pts.} \]

\textbf{Period-3 orbits?} i.e. \( f^3(x) = x, \quad x \in \mathbb{R} \)?

\[ f^3(x) = f(f^2(x)) = f(2x^2 - x^4) = 1 - (2x^2 - x^4)^2 = 1 - 4x^4 + 4x^4 - x^8 \]

\[ f^3(x) = x \quad \Rightarrow \]

\[ (1 + x^2 + x^3 - 2x^4 - x^5 + x^6)(1 - x - x^3) = 0 \]

\[ \text{no real roots, } x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \]

\textbf{Fixed pts.}

\[ \therefore f \text{ has no period-3 orbits.} \]

\textbf{NB!} In principle, one can find period-\( k \) pts by solving \( f^k(p) = p \).

But often not practicable, e.g. for \( f(x) \) quadratic, \( k = 10 \), \( f^{10}(x) \) 2\textsuperscript{5} = 1024 degree polynomials.
Sarkovskii's Theorem
— Sarkovskii Ordering

1st row 3 > 5 > 7 > 9 > 11 > ... all odd number m = 1

2nd row 6 > 10 > 14 > 18 > 22 > ... m = 2

3rd row 12 > 20 > 28 > 36 > 44 > ... m = 2^2

4th row 24 > 40 > 56 > 72 > 88 > ... m = 2^3

... m = 2^n

... 64 > 32 > 16 > 8 > 4 > 2 > 1

No: Any natural number can be expressed as \( 2^n \cdot q - 2^p \),
where \( q \) — odd integer, \( p \geq 0 \)

Properties.

1. Every natural number can be found exactly once.

in the Sarkovskii ordering. The order has a last

number — it is 1.

Recall: \( \rightarrow \) is also a

total order.

2. The relation \( \rightarrow \) is a total order on the natural numbers

i.e. 1. \( \forall j, k, \text{ either } j \rightarrow k, \text{ or } k \rightarrow j \)

2. \( j \rightarrow k, k \rightarrow j \Rightarrow j = k \)

3. \( j \rightarrow k, k \rightarrow l \Rightarrow j \rightarrow l \)
Sarkovskii's Theorem (1964)

Suppose that \( f \) is a continuous function of \( \mathbb{R} \), and that \( f \) has period-\( n \) pts, then \( f \) also has period-\( m \) pts if \( n \geq m \) is in Sarkovskii ordering.

\[ \text{No. 1: } 3 \Rightarrow 5, \forall j \in \mathbb{N}, \Rightarrow \text{ a continuous fn which has period-3 pts must have periodic pts of any periods.} \]

\[ \text{No. 2: if } f \text{ has a periodic orbit of period } k, \text{ and } k = 2^n, n = 0, 1, 2, \ldots \]

\[ \text{then } f \text{ must have an infinite number of periodic orbits} \]

\[ \text{(e.g. all orbits of period } 2^n, n = 0, 1, 2, \ldots) \]

\[ \text{No. 3: No differentiability is required for the function } f: \mathbb{R} \rightarrow \mathbb{R}, \]

\[ \text{No. 4: The theorem is sharp in that one can construct a} \]

\[ \text{continuous fn that has period-} n \text{ orbits and no periodic pts with period that precedes } n \]

\[ \text{in the Sarkovskii ordering.} \Rightarrow \]

A stronger version of Sarkovskii then:

If \( j \) appears before \( k \) in the ordering, then it is true that if a continuous fn \( f: \mathbb{R} \rightarrow \mathbb{R} \)

has period-\( j \) pts, then it must have period-\( k \) pts.
Otherwise (if \( j \) is often \( k \) in the ordering), it is false.

4. Due to the above stronger version,

\[ k \geq j \] — shorthand for the statement that

of a continuous function \( f: \mathbb{R} \rightarrow \mathbb{R} \) has a period-\( k \) point, then it must have period-\( j \) points.

**Hint:** Follow the outline in Challenge 3 (p. 135) in Chaos—an intro to Dyn. Systems by Alligood, Sauer, and Yorke.

**Examples.**

- **e.g.** Do all continuous functions which have period-5 pts have period-3 pts?

  No. \( (\because 5 \nmid 3 \text{ is false}) \)

- **e.g.** 2 \( \nmid 1 \) means “Continuous fn of \( f: \mathbb{R} \rightarrow \mathbb{R} \) with period-2 pts must have fixed pts” — obvious!

- **e.g.** 1 \( \nmid 2 \) is not true e.g. \( f(x) = x^2 \) has fixed pts, but no period-2 pts.
The Logistic Map \( G(x) = 4x(1-x) \)

Q. How many periodic orbits are there?

\[
G(x) = \begin{cases} 
1 \text{ bump} & \quad \text{for } G^2(x) = 0, \quad G^2(a_1) = G^2(a_2) = \frac{1}{2} \\
2 \text{ bumps} & \quad \text{for } G^2(x) = G^2(a_0) = 1 \Rightarrow 2 \text{ bumps} \\
4 \text{ bumps} & \quad \text{for } G^2(x) = \frac{1}{2}, \quad G^2(b_1) = 1 \Rightarrow 4 \text{ bumps} \\
\end{cases}
\]

\[G(x) : \quad 2 \text{ fixed pts.}\]
\[G^2(x) : \quad 4 \text{ fixed pts, 2 of which are fixed pts of } G(x)\]
\[G^4(x) : \quad 8 \text{ fixed pts, 2 of which are fixed pts of } G^2(x) \]

\[\therefore G^2(p_1) = p_1, \quad G^2(p_2) = G(p_1) = p_2 \]
\[\Rightarrow \quad p_1, p_2 \text{ are fixed pts of } G^3. \]
\[\therefore \quad 6 \text{ period-3 pts} \quad \Rightarrow \quad 2 \text{ period-3 orbits}. \]

\[\therefore \quad G(x) \text{ has periodic orbits of any period.} \quad (\text{Sakowskii Thm})\]
The theorem can be used to show that certain periods do not exist.

\[ f = 3.2x(1-x) \]

**fixed pt:** \( x_1 = 0, \ x_2 = 0.7 \)

**period-2 pts:** \( x_3 \approx 0.5, \ x_4 \approx 0.8 \)

It does not have higher periods, it does not have period-4 or 5.

\( \therefore \ 4 > 2 \times 1 \) (the end of Sharkovskii ordering)

\( \therefore \ f \) has no periods except \( 2 \) and \( 1 \).
Stability of Periodic Orbits

**Def.** \( p \) — a period-\( k \) pts of \( f \).

The period-\( k \) orbits of \( p \) is a periodic sink (source) if \( p \) is a sink (source) for the map \( f^k \).

\[ (f^{2k})'(x) = \left[ f(f(x))' \right]' = f'(f(x)) f'(x) \]

\[ \Rightarrow p, p_2 \text{ period-2, then } p_2 = f^{2k}p. \]

\[ \therefore (f^{2k})'(p) = f'(p_2) \cdot f'(p) \]

\[ \therefore (f_{2k})'(p) = f'(p_2) \cdot f'(p) \]

\[ (f^{2k})'(p_i) = (f^{2k})'(p) \]

\[ \text{— same for every pt in the periodic orbit.} \]

\[ \therefore \text{ makes sense to speak about the stability of a period-} k \text{ orbit.} \]

**Thus.** The periodic orbit \( p, p_2 \cdots p_k \) is a sink (source) if \( |f'(p)\cdots f'(p_k)| < 1 \) (\( > 1 \)).

**Note.** Stability of a periodic orbit is a collective property, i.e., determined by every periodic pt on the orbit.
Periodic orbits of $\sigma(x)$

(i). Fixed pts of $\sigma(x)$, i.e. $\sigma(x) = x$

\[\sigma(0, a_1 a_2 a_3 \ldots) = 0.a_2 a_3 \ldots = x = 0.a_1 a_2 \ldots\]

.: $a_1 = a_2 = a_3 = a_4 = \ldots$

.: $a_i = 0$ or $a_i = 1$ \( \forall i \)

$0.1 = 1$, or $0$ are fixed pts.

No, usually $x = 1$ is excluded

in the def of $\sigma(x)$.

(ii). Period-2 orbits:

\[x = 0.a_1 a_2 a_3 \ldots\]

\[\sigma(x) = 0.a_2 a_3 a_4 \ldots\]

\[\sigma^2(x) = 0.a_3 a_4 a_5 \ldots\]

\[\sigma^2(x) = x \rightarrow a_1 = a_3 = a_5 = \ldots\]

\[a_2 = a_4 = a_6 = \ldots\]

.: Add all $0$ or all $1$

0-over all $0$ or all $1$.

.: Periodic orbits are

\[0.000\ldots = 0 \text{ -- fixed pt}\]

\[0.0101\ldots = \frac{1}{3} \text{ -- period-2.}\]

\[0.1010\ldots = \frac{2}{3}\]

.: $\sigma\left(\frac{1}{3}\right) = \frac{2}{3}$, $\sigma\left(\frac{2}{3}\right) = \frac{1}{3}$.
iii) Period-3 orbits. \( 0^3(x) = x \)

\[
\begin{align*}
&\Rightarrow \frac{1}{3} \rightarrow \frac{2}{7} \rightarrow \frac{6}{7} \rightarrow \frac{1}{7} \\
&\frac{3}{7} \rightarrow \frac{6}{7} \rightarrow \frac{5}{7} \rightarrow \frac{3}{7} \quad \text{etc.}
\end{align*}
\]

\[\text{Note: In the binary representations, one can immediately see which pts in } [0,1) \text{ are periodic — these are the pts of a repeating binary expansion.}\]

\( \text{e.g. } \frac{1}{5} = (.0011\overline{0011})_2 \)

\[
\begin{align*}
&\sigma^{0}(.\overline{1}) = (.011\overline{0011})_2 \\
&\sigma^{1}(.\overline{1}) = (.111\overline{0011})_2 \\
&\sigma^{2}(.\overline{1}) = (.1\overline{0011})_2 \\
&\sigma^{3}(.\overline{1}) = (.0\overline{0011})_2
\end{align*}
\]

\[\Rightarrow x_{\frac{1}{5}} \text{ is a period-4 pt.}\]

\[\text{Note: All periodic orbits of } \sigma(x) = 2x \mod 1 \text{ are unstable.}\]

\[
\left| \sigma^{n}(x) \right| = 2 \quad \forall x
\]

\[\text{i.e. Lyapunov exponent } \lambda_{p} = \ln 2 > 0 \text{ for periodic orbits.}\]
6. For Bernoulli shift, pts on periodic orbits are dense in $[0,1)$ i.e. $\forall x \in [0,1)$, $\forall \varepsilon > 0 \exists$ at least one pt on a periodic orbit in $N^k(\varepsilon)$. (as a matter of fact, there are an infinite # of such pts)

In interval $[\frac{m-1}{2^k}, \frac{m}{2^k})$, $m=1,2,\ldots,2^k$ there is one fixed pt of $\sigma^k(x)$

$\therefore$ there exists at least one periodic pt of $\sigma$ in $N^k(\varepsilon)$ for $2^{-k} < \varepsilon$.

$\Rightarrow$ Periodic pts are dense for $\sigma^k(x)$.

6. The periodic pts are countably infinite. all pts in $[0,1)$ — uncountable.

$\Rightarrow$ Periodic orbits dense for shift, therefore

$\therefore$ Randomly pick up a pt in $[0,1)$ uniformly,

$\Rightarrow$ most likely not a periodic orbit.
Implications:

1. Nonperiodic orbits are typical \( \forall x \in [0,1] \) — natural min. meas.

2. Periodic orbits are unstable, small errors lead away from periodic orbits
to typical orbits \( \Rightarrow \) natural min. meas.

\( \exists \sigma \) = 1

3. If \( x_0 \) is chosen exactly to be a periodic pt.
\( \Rightarrow \) the orbit generates a singular meas.

But this set of initial pts \( \Rightarrow \) meas 0.

P.S.

1. How many period 4 orbits are there for the Bernoulli Shift?

How many period 4 orbits are there for the map \( x_{n+1} = 3x_n \mod 1 \)?
**Sharkovskii’s Theorem**

A continuous map of the unit interval $[0, 1]$ may have one fixed point and no other periodic orbits (for example, $f(x) = x/2$). There may be fixed points, period-two orbits, and no other periodic orbits (for example, $f(x) = 1 - x$). (Recall that $f$ has a point of period $p$ if $f^p(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < p$.)

As we saw in Exercise T3.10, however, the existence of a periodic orbit of period three, in addition to implying sensitive dependence on initial conditions (Challenge 1, Chapter 1), implied the existence of orbits of all periods. We found that this fact was a consequence of our symbolic description of itineraries using transition graphs.

If we follow the logic used in the period-three case a little further, we can prove a more general theorem about the existence of periodic points for a map on a one-dimensional interval. For example, although the existence of a period-5 orbit may not imply the existence of a period-3 orbit, it does imply orbits of all other periods.

Sharkovskii's Theorem gives a scheme for ordering the natural numbers in an unusual way so that for each natural number $n$, the existence of a period-$n$ point implies the existence of periodic orbits of all the periods higher in the ordering than $n$. Here is Sharkovskii’s ordering:

$$3 < 5 < 7 < 9 < \ldots < 2 \cdot 3 < 2 \cdot 5 < \ldots < 2^2 \cdot 3 < 2^2 \cdot 5 < \ldots$$

$$\ldots < 2^3 \cdot 3 < 2^3 \cdot 5 < \ldots < 2^4 \cdot 3 < 2^4 \cdot 5 < \ldots < 2^3 < 2^2 < 2 < 1.$$

**Theorem 3.31** Assume that $f$ is a continuous map on an interval and has a period $p$ orbit. If $p < q$, then $f$ has a period-$q$ orbit.

Thus, the existence of a period-eight orbit implies the existence of at least one period-four orbit, at least one period-two orbit, and at least one fixed point. The existence of a periodic orbit whose period is not a power of two implies the existence of orbits of all periods that are powers of two. Since three is the “smallest” natural number in the Sharkovskii ordering, the existence of a period-three orbit implies the existence of all orbits of all other periods.

The simplest fact expressed by this ordering is that if $f$ has a period-two orbit, then $f$ has a period-one orbit. We will run through the reason for this fact, as it will be the prototype for the arguments needed to prove Sharkovskii’s Theorem.
Let $x_1$ and $x_2 = f(x_1)$ be the two points of the period-two orbit. Since $f(x_1) = x_2$ and $f(x_2) = x_1$, the continuity of $f$ implies that the set $f([x_1, x_2])$ contains $[x_1, x_2]$. (Sketch a rough graph of $f$ to confirm this.) By Theorem 3.17, the map $f$ has a fixed point in $[x_1, x_2]$.

The proof of Sharkovskii's theorem follows in outline form. We adopt the general line of reasoning of Block et al. 1979. In each part, you are expected to fill in an explanation. Your goal is to prove as many of the propositions as possible.

Assume $f$ has a period $p$ orbit for $p \geq 3$. This means that there is an $x_i$ such that $f^n(x_i) = x_i$ holds for $n = p$ but not for any other $n$ smaller than $p$. Let $x_1 < \cdots < x_p$ be the periodic orbit points. Then $f(x_1)$ is one of the $x_i$, but we do not know which one. We only know that the map $f$ permutes the $x_i$. In turn, the $x_i$ divide the interval $[a, b] = [x_1, x_p]$ into $p - 1$ subintervals $[x_1, x_2], [x_2, x_3], \ldots, [x_{p-1}, x_p]$. Note that the image of each of these subintervals contains others of the subintervals. We can form a transition graph with these $p - 1$ subintervals, and form itineraries using $p - 1$ symbols.

Let $A_1$ be the rightmost subinterval whose left endpoint maps to the right of itself. Then $f(A_1)$ contains $A_1$ (see Figure 3.14 for an illustration of the $p = 9$ case).

**Step 1** Recall that the image of an interval under a continuous map is an interval. Use the fact that $A \subseteq B \Rightarrow f(A) \subseteq f(B)$ to show that

$$A_1 \subseteq f(A_1) \subseteq f^2(A_1) \subseteq \ldots.$$ 

(We will say that the subintervals $A_1, f(A_1), f^2(A_1), \ldots$ form an increasing "chain" of subintervals.)

**Step 2** Show that the number of orbit points $x_i$ lying in $f(A_1)$ is strictly increasing with $j$ until all $p$ points are contained in $f^k(A_1)$ for a certain $k$. Explain why $f^k(A_1)$ contains $[x_1, x_p]$. Use the important facts that the endpoints of each

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**Figure 3.14** Definition of $A_1$.

$A_1$ is chosen to be the rightmost subinterval whose left-hand endpoint maps to the right under $f$. 

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subinterval are obliged to map to the subinterval endpoints from the partition, and that each endpoint must traverse the entire period-$p$ orbit under $f$.

As a consequence of Step 2, the endpoints of $A_1$ cannot simply map among themselves under $f$—there must be a new orbit point included in $f(A_1)$, since $p \neq 2$. So at least one endpoint maps away from the boundary of $A_1$, implying that $f(A_1)$ contains not only $A_1$ but another subinterval, which we could call $A_2$.

**Step 3** Prove that either (1) there is another subinterval (besides $A_1$) whose image contains $A_1$, or (2) $p$ is even and $f$ has a period-two orbit. [Hints: By definition of period, there are no periodic orbits of period less than $p$ among the $x_i$. Therefore, if $p$ is odd, the $x_i$ on the "odd" side of $A_1$ cannot map entirely amongst themselves, and cannot simply exchange points with the "even" side of $A_1$ for arithmetic reasons. So for some subinterval other than $A_1$, one endpoint must be mapped to the odd side of $A_1$ and the other to the even side. If $p$ is even, the same argument shows that either there is another subinterval whose image contains $A_1$, or else the $x_i$ on the left of $A_1$ map entirely to the $x_i$ on the right of $A_1$, and vice versa. In this case the interval consisting of all points to the left of $A_1$ maps to itself under $f^2$.]

**Step 4** Prove that either (1) $f$ has a periodic orbit of period $p - 2$ in $[x_1, x_p]$, (2) $p$ is even and $f$ has a period-two orbit, or (3) $k = p - 2$. Alternative (3) means that $f(A_1)$ contains $A_1$ and one other interval from the partition called $A_2$, $f^2(A_1)$ contains those two and precisely one more interval called $A_3$, and so on. [Hint: If $k \leq p - 3$, use Step 3 and the Fixed-Point Theorem (Theorem 3.17) to show that there is a length $p - 2$ orbit beginning in $A_i$.]

Now assume that $p$ is the smallest odd period greater than one for which $f$ has a periodic orbit. Steps 5 and 6 treat the concrete example case $p = 9$.

**Step 5** Beginning with Figure 3.14, show that the endpoints of subintervals $A_1, \ldots, A_8$ map as in Figure 3.15, or as its mirror image. Conclude that $A_1 \subseteq f(A_8)$, and that the transition graph is as shown in Figure 3.16 for the itineraries of $f$. In particular, $A_8$ maps over $A_i$ for all odd $i$.

**Step 6** Using symbol sequences constructed from Figure 3.16, prove the existence of periodic points of the following periods:

(a) Even numbers less than 9;
(b) All numbers greater than 9;
(c) Period 1.

This proves Sharkovskii's Theorem for maps where 9 is the smallest odd period.
Figure 3.15 A map with a period-nine orbit and no period-three, -five, or -seven orbits.
It must map subintervals as shown, or as the mirror image of this picture.

Step 7 In Steps 5 and 6, we assumed that the smallest odd period was 9. Explain how to generalize the proof from 9 to any odd number greater than 1. Note that Step 4 is not required for the $p = 3$ case.

Step 8 Prove that if $f$ has a periodic orbit of even period, then $f$ has a periodic orbit of period-two. [Hint: Let $p$ be the smallest even period of $f$. Either Step 3 gives a period-two orbit immediately, or Step 4 applies, in which case Steps 5 and 6 can be redone with $p$ even to get a period-two orbit.]

Figure 3.16 Transition graph for period-nine map.
The existence of orbits of periods 1, 2, 4, 6, 8, and all numbers greater than 9 is implied by this graph.
Step 9  Prove that if \( f \) has a periodic orbit of period \( 2^k \), then \( f \) has periodic orbits of periods \( 2^{k-1}, \ldots, 4, 2, 1 \). (Since \( f^{2^k-2} \) has a period-four point, it has a period-two point, by Step 8. Ascertain the period of this orbit as an orbit of \( f \).)

Step 10  Assume \( p = 2^k q \) is the leftmost number on the list for which \( f \) has a period-\( p \) point, where \( q \) is an odd number greater than 1. The integers to the right of \( p \) in the list are of two types: either \( 2^k r \), where \( r \) is greater than \( q \), or an integer power of 2. Since \( f^{2^k} \) has a period-\( q \) orbit, Step 7 implies that \( f^{2^k} \) has orbits of every period \( r \) greater than \( q \). Our choice of \( p \) implies that these orbits are not orbits of \( f \) whose period divides evenly into \( p \). Use these periodic orbits and Step 9 to complete the proof of Sharkovskii’s Theorem.