Calculus III

Lecture 1: Three-Dimensional Coordinate Systems

* In 1 dimension, only 1 number is necessary to locate a point. One chooses an origin on the one-dimensional line, and locate the point with respect to the origin:

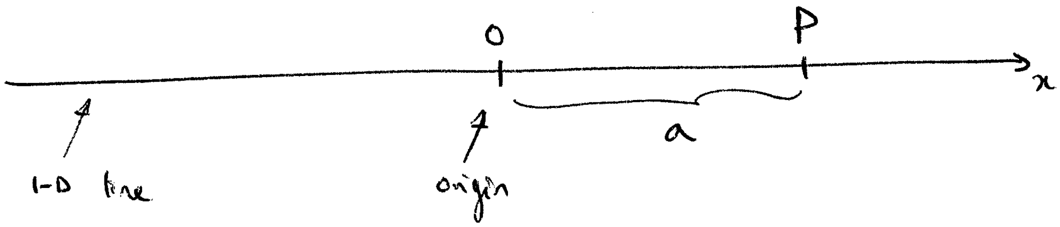


Figure 1.1

The real number a uniquely determines the position of the point P along the line

* In 2 dimensions, two numbers are necessary to locate a point. We introduce two axes, i.e. the x-axis and the y-axis, together with an origin O, as follows:

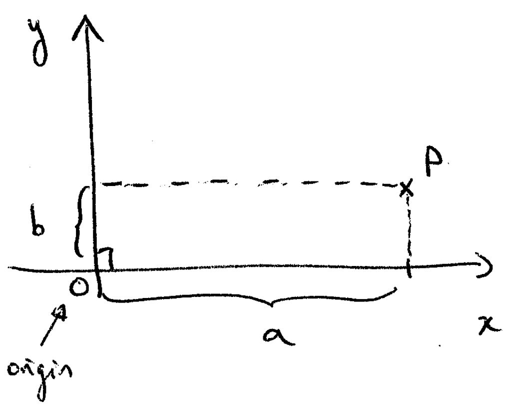


Figure 1.2

With these axes, any point P in the two-dimensional plane is uniquely given by the ordered pair (a,b) of real numbers, where a is the x-coordinate of the point P, and b is the y-coordinate of the point P.

* To locate a point in <u>3 dimensions</u> (i.e. in space), <u>three numbers</u> are necessary. Building on the previous case, to locate and represent a point in space, we introduce <u>3 axes</u> (the coordinate axes), and an origin as follows:

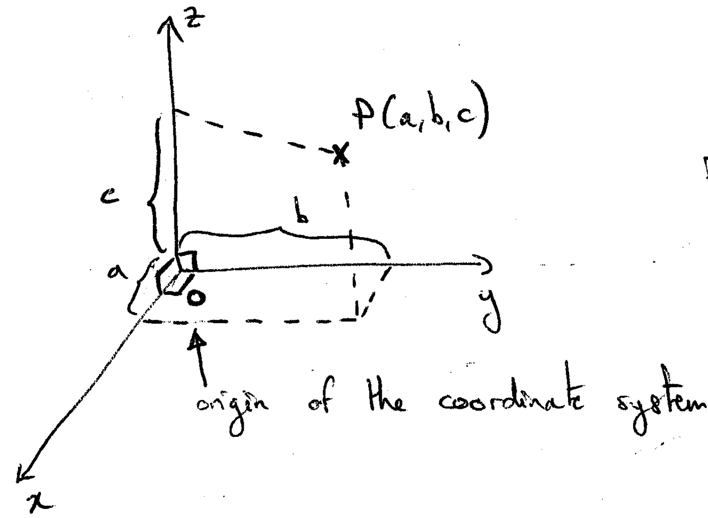


Figure 1.3

The axes are all <u>perpendicular</u> to each other (as shown by the ⌉ symbols on the figure above): the x-axis is perpendicular to the y-axis and the z-axis, and the y-axis is perpendicular to the z-axis (and vice-versa for all these statements).

Note that we could have imagined constructing the three axes in such a way as to have the z-axis pointing down:

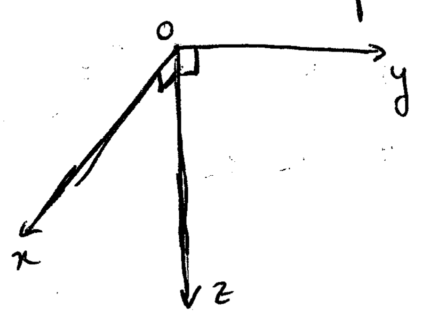


Figure 1.4

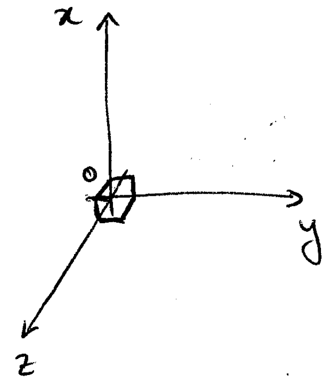
DOES NOT SATISFY RIGHT-HAND RULE

The reason we chose the z-axis to point upwards is that in this configuration, the coordinate axes follow the <u>right-hand rule</u>: if you place your RH thumb along the x-axis, and your RH index along the y-axis, your

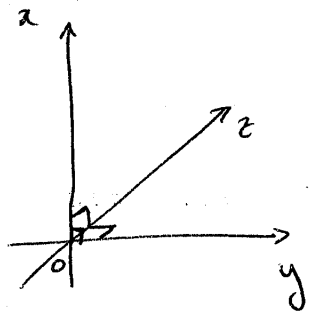
RH middle finger gives the direction of the z-axis.
In this class, we will <u>always</u> use coordinate systems that <u>follow the right-hand rule</u>.

<u>QUESTION</u>: Which of the following coordinate axes are in a configuration that satisfies the right-hand rule?

a)



b)



The three axes x, y and z in Figure 1.3 define three planes: <u>the coordinate planes</u>. The xy-plane contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes.

With these planes, the ordered triple (a, b, c) uniquely determining the location of the point P in space can be given a geometrical interpretation:

- a, the x-coordinate of P, is the distance from the yz-plane to P (along the direction of the x-axis)
- b, the y-coordinate of P, is the distance from the xz-plane to P (along the direction of the y-axis)
- c, the z-coordinate of P, is the distance from the xy-plane to P (along the direction of the z-axis)

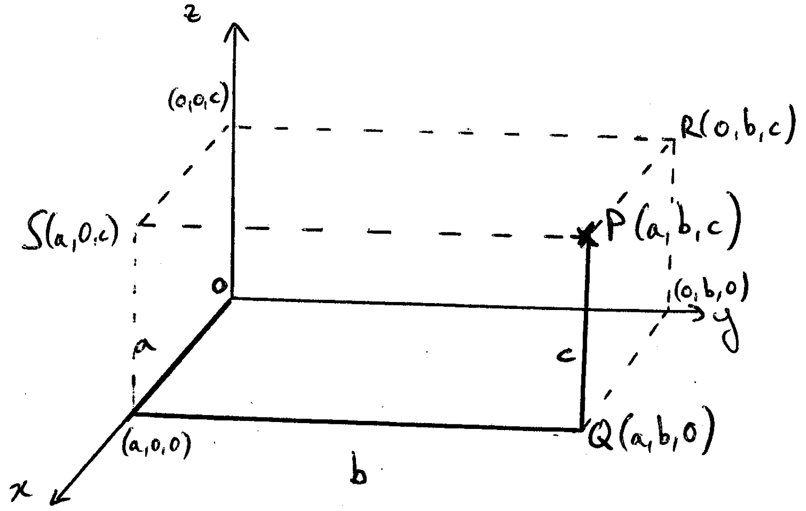


Figure 1.5

The point Q with coordinates $(a, b, 0)$ is called the projection of P on the xy-plane.
The point R with coordinates $(0, b, c)$ is the projection of P on the yz-plane; S with coordinates $(a, 0, c)$ is the projection of P on the xz-plane.

QUESTION: Can you locate and plot the point $P(-1, 3, 1)$?

Any ordered triple (a,b,c) of real numbers a, b and c is uniquely associated with a point in space, and any point in space can be described by a unique ordered triple (a,b,c) with a, b, and c real numbers. In mathematical terms, we say that there is a one-to-one correspondence between points in space and the set $\{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

symbol for "such that"

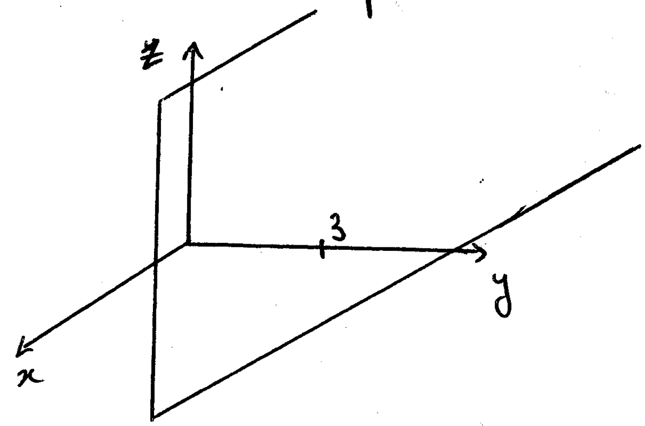
The ~~space~~ set $\{(a,b,c) \mid a,b,c \in \mathbb{R}\}$ is often written $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$

By associating any point in space with the ordered triple $(a,b,c) \in \mathbb{R}^3$, where a is the x-coordinate, b is the y-coordinate, and the z-coordinate, we have defined a three-dimensional rectangular coordinate system.

We can look at regions in three-dimensional space defined by particular subsets of $\mathbb{R}^3$, and consider for example equations of planes:

The equation $y=3$ corresponds to the following subset of $\mathbb{R}^3$: $\{x,y,z \mid y=3\}$, i.e all the points in space which have 3 as the y-coordinate. This is the vertical plane parallel to the xz-plane and three units in front of it:

Figure 1.6

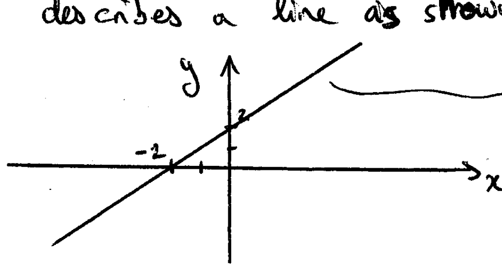


Note that while in Figure 1.6 the plane appears limited in extent in the x-direction and the z-direction, this is only for visualization purposes. In reality, the plane $y=3$ is infinite in extent in both the x-direction and the z-direction.

<u>QUESTION</u>: Describe and sketch the plane in $\mathbb{R}^3$ represented by the equation $y = x+2$

<u>NOTE</u>: When an equation is given that involves only two variables (e.g. x and y as in the previous example), only the context can tell us if one is looking for a surface in three dimensions or a curve in two dimensions. For example, in $\mathbb{R}^2$ the equation $y = x+2$ describes a line as shown in the graph below:

Figure 1.7



In $\mathbb{R}^3$, $y = x+2$ describes a plane, as you found answering the question above.

DISTANCE

Now that we know how to locate and represent points in three dimensions, we might want to know how to measure the distance between two points P_1 and P_2.

In two dimensions, we know that the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, written $|P_1P_2|$, is given by the formula:

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

It is easy to extend this formula to the three-dimensional case:

DISTANCE FORMULA IN THREE DIMENSIONS:

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Proof: To prove this formula, we only need two tools: the formula for the distance in two dimensions and the Pythagorean theorem.

We start by locating the two points P_1 and P_2 in our coordinate axes:

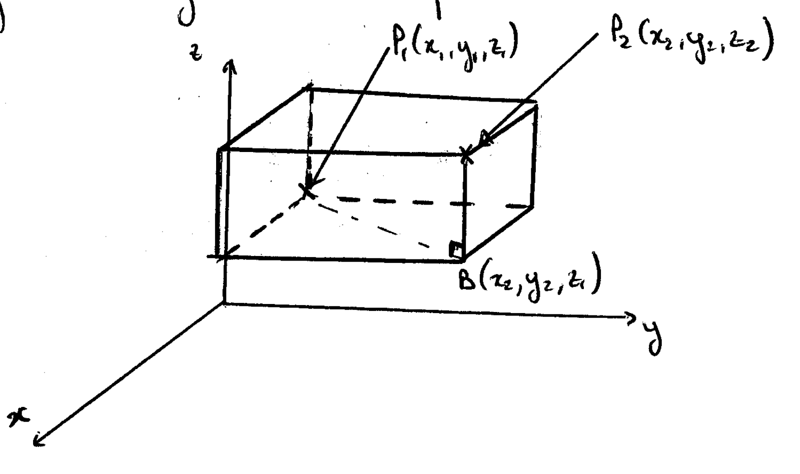


Figure 1.8

We then construct a rectangular box where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. Consider the vertex $B(x_2, y_2, z_1)$ of the box, which is the projection of P_2 on the plane with equation $z = z_1$. Because the box is rectangular, the triangle $P_1 B P_2$ is a <u>right-angled triangle</u>. We can therefore apply the Pythagorean theorem:

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

The points P_1 and B are in the same plane $z = z_1$, so we can use the formula for the distance in two dimensions to find

$$|P_1B| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \Rightarrow |P_1B|^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

The points P_2 and B are in the same plane $x = x_2$, so we can use the formula for the distance in two dimensions to find

$$|BP_2| = \sqrt{(y_2-y_2)^2 + (z_2-z_1)^2} = |z_2-z_1| \Rightarrow |BP_2|^2 = (z_2-z_1)^2$$

Combining the two results, we therefore find

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$$

$$\Rightarrow |P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \quad \blacksquare$$

<u>QUESTION:</u> Compute the distance from the point $P(-2,1,1)$ to the point $Q(1,3,-2)$

Now that we are able to compute the distance between points in space, we can look at another type of surface in three dimensions: the sphere.

Given a point $C(a,b,c)$, a sphere with center C and radius R is the set of points $P(x,y,z)$ satisfying $|PC| = R$. Let us look at the set of elements (x,y,z) in $\mathbb{R}^3$ that satisfy this property:

$$|PC| = R \iff \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = R$$

$$\boxed{\iff (x-a)^2+(y-b)^2+(z-c)^2 = R^2}$$

We have just derived the generic equation of a sphere of center (a,b,c) and radius R:

EQUATION OF A SPHERE:

An equation of a sphere with center $C(a,b,c)$ and radius R is

$$(x-a)^2+(y-b)^2+(z-c)^2 = R^2$$

In particular, if the center is the origin O, then an equation of the sphere of radius R is $\quad x^2+y^2+z^2 = R^2$

QUESTION: Show that $x^2+y^2+z^2-6x+2y-10z+31=0$ is the equation of a sphere, and find its center and radius

QUESTION: What region in $\mathbb{R}^3$ is represented by the following inequality:

$$(x-3)^2+y^2+z^2 \geqslant 3$$