Calculus III

Lecture 10: Functions of several variables

I/ Functions of two variables

Starting from this lecture, we switch gear and look at a new topic: functions of several variables. We will look at ways to represent them, take limits with these functions, take derivatives, integrate...

The link between this topic and all the previous lectures on geometry in space will become apparent when we try to plot the functions of several variables, and want to describe/analyze the graphs.

1) Background - Introduction

In previous courses, we have studied functions of one variable: to one given input x, the function f assigns the number $f(x)$. For example, the function sin assigns the number $\sin x$ to any real input x

A function of two variables can be easily understood in that light. It is a function which takes two inputs and assigns a number to these two inputs.

Example: $f(x,y) = x^2 + y^2$

To the input $(3,2)$, $f(x,y)$ assigns $f(3,2) = 3^2 + 2^2 = 13$

2) Definition

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$.
The set D is the domain of f, and its range is the set of values that f takes on, i.e $\{f(x,y) \mid (x,y) \in D\}$

QUESTION: What is the domain of the function $f(x,y) = x^2 + y^2$?

QUESTION: What is the domain of the function $f(x,y) = \frac{1}{x+y}$?

3) Why are functions of two variables useful?

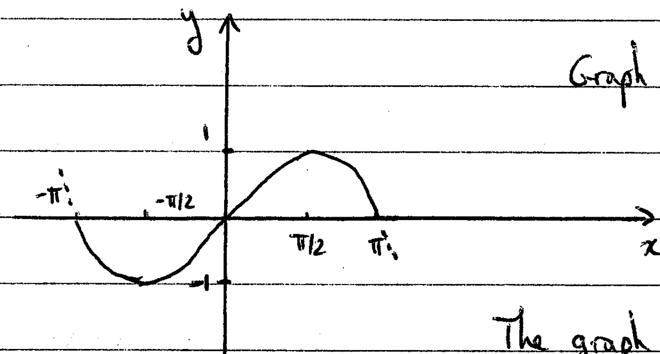
Functions of two variables are a very natural and useful tool in a large number of fields, such as geography, meteorology, economics, etc.

A very common example is the function giving us the temperature at every point on Earth. The function takes two inputs: the latitude and the longitude, and spits out a number, namely the temperature, for any of these two inputs.

Topographic maps are another good example: for any point given by the coordinates (x,y), the map tells us the altitude $h(x,y)$.

II] Graphs: how do we plot functions of two variables?

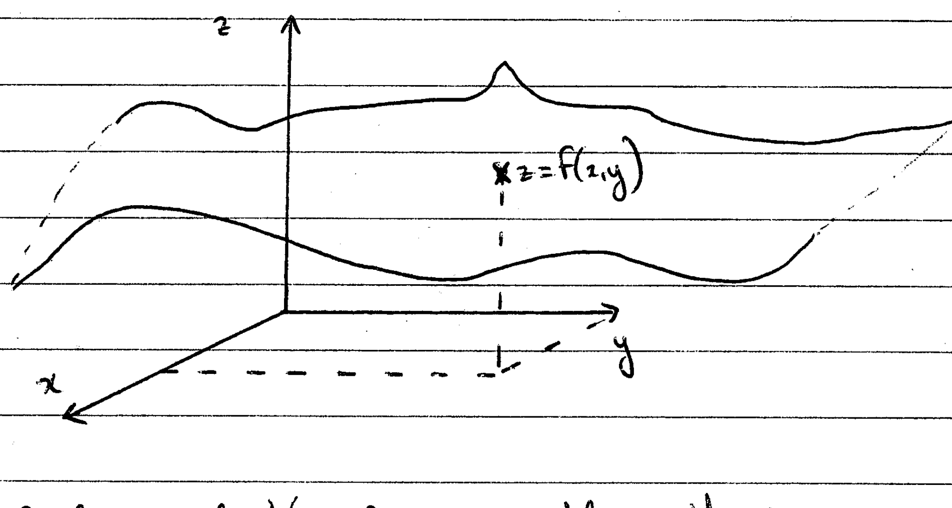
For a function of one variable, we know that a graph is constructed by taking x and y axes, and for each x, plot the point with coordinates $(x, y = f(x))$



Graph of $y = \sin x$

The graph of a function of one variable is a curve

For functions of two variables, we adopt the same idea. Since we now have two independent inputs x and y, we need three axes: x, y and z. For each pair (x,y), we then plot the point with coordinates $(x, y, z = f(x,y))$. This is how we obtain the graph of the function!



The graph of a function of two variables is a surface

Definition: If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in D.

QUESTION: Sketch the graph of the function $f(x,y) = x$

QUESTION: Sketch the graph of the function $f(x,y) = x^2 - y^2$

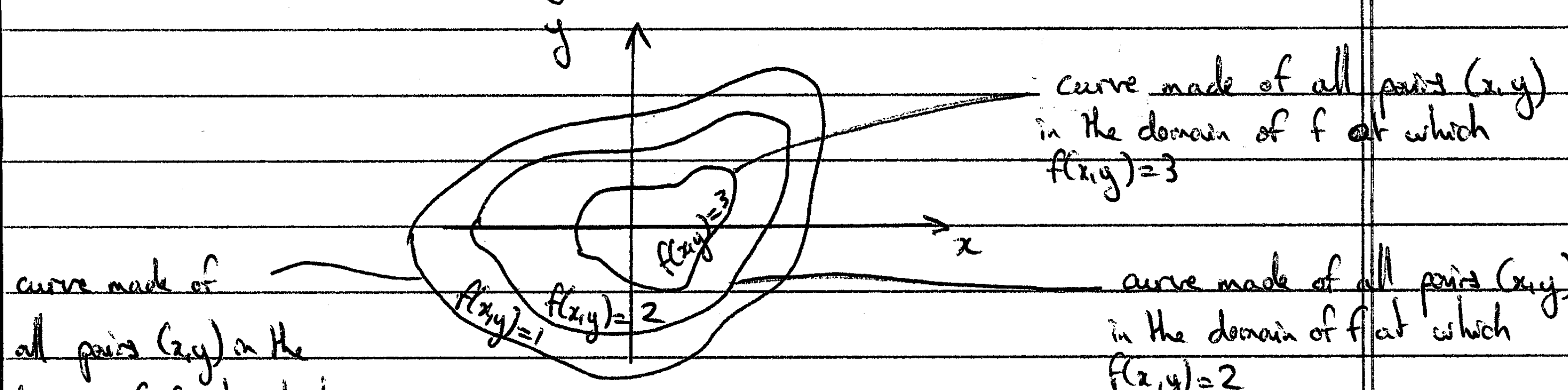
III] Level curves (aka Contour Plots)

1) Definition

As we have already noticed in Lecture 6, it can be quite hard to draw and sometimes even to read and analyze surfaces in space. Because of this, people have come up with a different, equivalent way of representing functions of two variables: it is called level curves or contour plots

Definition: The level curves of a function f of two variables are the curves with equations $f(x,y) = k$, where k is a constant in the range of f.

Often, when plotting contours, we take k evenly spaced:



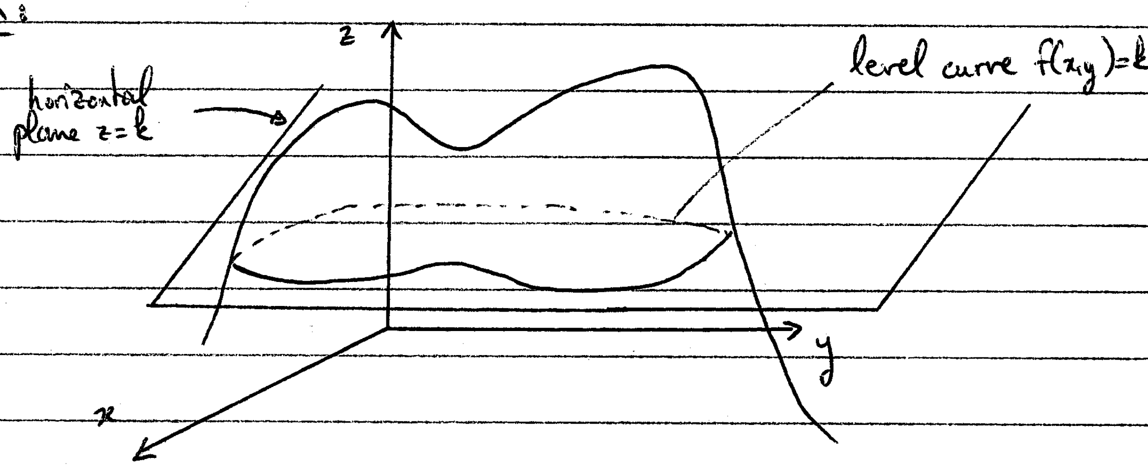
Level curves can be seen as a plot showing us the lines on which the height of the graph of f (i.e. its z value) is k.

Contour plots are quite handy in that for a given x and y, we can readily read off the <u>approximate value</u> of $f(x,y)$ from the plot. It is often easier to use as the graph of f in 3D space.

2) <u>Relationship between level curves and graphs</u>

If we slice the graph of f by the horizontal plane $z=k$, we obtain the level curve $f(x,y)=k$.

<u>Illustration:</u>



By repeating the process for several different k, we obtain the full contour plot of f

<u>QUESTION</u>: Sketch the level curves of $f(x,y)=x$

<u>QUESTION</u>: Give the contour plot of $f(x,y)=x^2-y^2$

3) Generalization: functions of 3 variables, functions of n variables

A function of three variables assigns a unique real number $f(x,y,z)$ to each ordered triple (x,y,z) in its domain.

Example: The temperature does not only depend on longitude and latitude, but also on altitude. We may write $T(x,y,z)$

More generally, a function of n variables assigns a unique real number $f(x_1, x_2, \ldots x_n)$ to each ordered n-tuples $(x_1, x_2, \ldots, x_n)$ in its domain.

Functions of three variables or more cannot really be represented with a graph, since that would require a space of four dimensions (for three variables) or more (for more than three variables).

For three variables, we still can obtain good insights about $f(x,y,z)$ by using contour plots. For functions of two variables, there were sets of level curves. For functions of three variables, they are sets of level surfaces.

The level surfaces are given by the equations $f(x,y,z)=k$ for k constant.

QUESTION: Find the level surfaces of $f(x,y,z)=x^2+y^2+z^2$