Calculus III

Lecture 19: Double integrals in polar coordinates

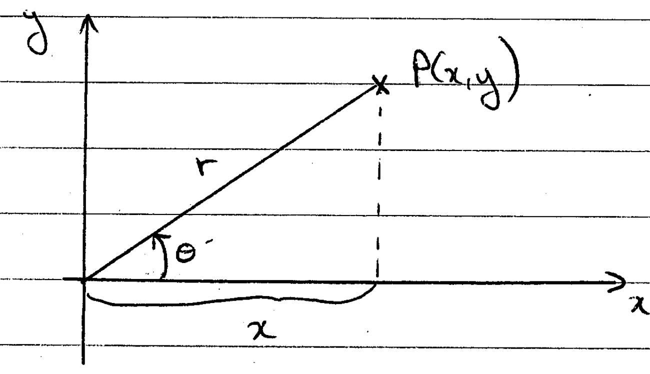
We have seen in previous courses that by appropriately choosing the variable in which a function f is express, it can be easier to calculate the integral of f. This is true for double integrals as well.

For double integrals, there can be another justification for choosing different variables to calculate the integrals: the region over which one calculates the integral may be more easily expressed in terms of the new variables, and the limits of integration in the iterated integrals may be simpler.

The purpose of this lecture is to see how polar coordinates can simplify the calculation of double integrals over certain regions.

1) Short review of polar coordinates

Let us consider a point $P(x,y)$ in the x-y plane:



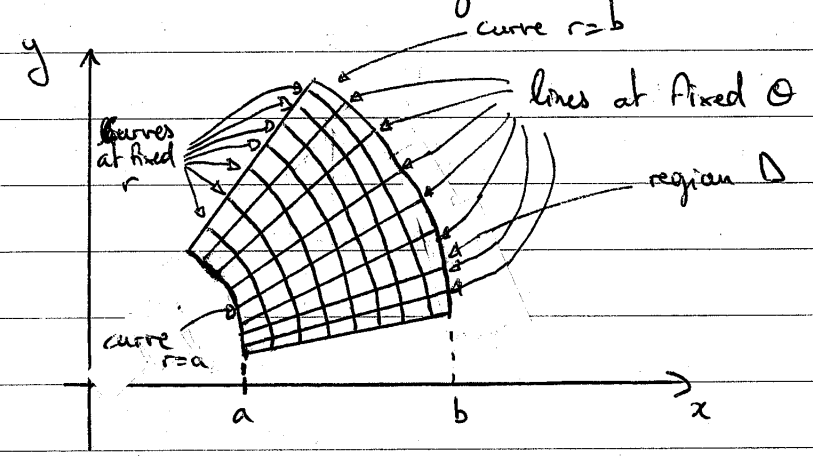
The point P can also be written $P(r,\theta)$, i.e in terms of the polar coordinates (r,θ) defined in terms of x and y as follows:

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\theta \quad y = r\sin\theta$$

Note that $r \geqslant 0$, and θ is measure from the x-axis.
$\theta \in [0, 2\pi[$

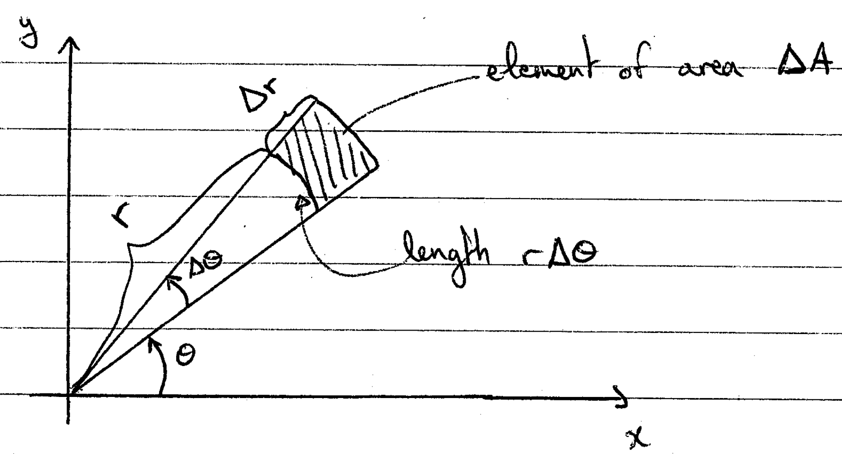
2) Element of area in polar coordinates

Let us now consider a region D in the x-y plane:



We can decompose this region in small elements of area in polar coordinates, i.e using lines at fixed θ and curves at fixed r.

Let us now focus on one of the small elements of area ΔA and look at how it can be expressed in terms of r, θ, Δr and $\Delta\theta$:



In the limit of Δr and $\Delta\theta$ small, the curvature of the element vanishes, and the area ΔA is that of a rectangle.

- Two of the sides of the rectangle have length Δr.
- The other two sides of the rectangle have length $r\Delta\theta$ (indeed, the perimeter of a circle is $2\pi r$, and here we do not go the full 2π in angle, but just $\Delta\theta$)

Thus, the element of area is : $\Delta A = \Delta r \, r\Delta\theta = r\Delta r\Delta\theta$

In the limit of very small $\Delta r = dr$ and $\Delta\theta = d\theta$, we write:

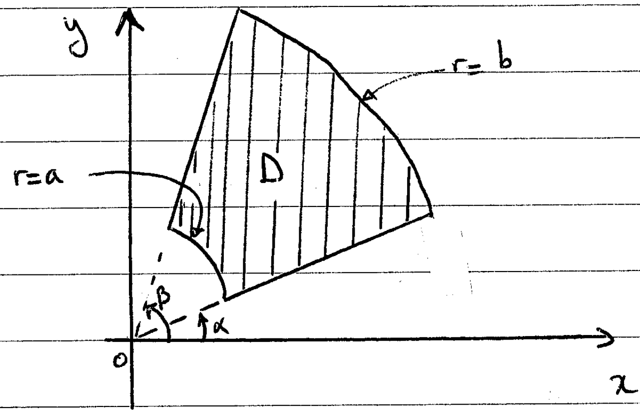
$$\boxed{dA = r\,dr\,d\theta}$$

⚠ <u>WARNING</u> : It is important not to forget the r multiplying $dr\,d\theta$ in the previous expression

3) <u>Double integrals in polar coordinates</u>

We now have all the tools we need to express double integrals in polar coordinates. Consider a region D given by:

$$D = \{(r,\theta) \mid 0 \le a \le r \le b, \ \alpha \le \theta \le \beta\}$$



D is often called a <u>polar rectangle</u> because the value of r on the boundary does not depend on θ, and the value of θ on the boundary does not depend on r.

At θ fixed, the region D goes from $r=a$ to $r=b$
At r fixed, the region D goes from $\theta=\alpha$ to $\theta=\beta$

Thus, the double integral of a function f over D is:

$$\boxed{\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

<u>Note:</u> In principle, the order of integration can be exchanged between r and θ. However, in the large majority of cases, it will be more convenient to calculate the integral with respect to r first.

<u>**QUESTION:**</u> Give an expression in (x,y) coordinates of the integral of $f(x,y) = 1-x^2-y^2$ over the quarter disk $x^2+y^2 \leq 1,\ x\geq 0,\ y \geq 0$

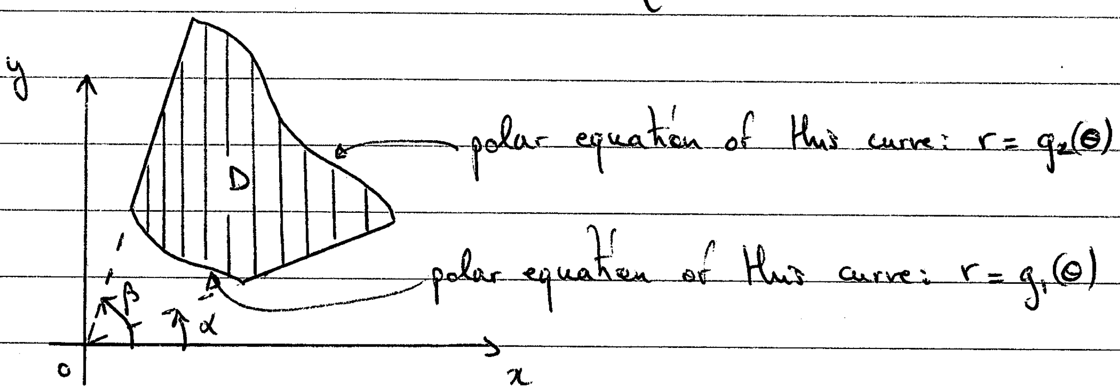
Give an expression of the same double integral in polar coordinates, and calculate the integral.

If you did everything right, the expression in polar coordinates should be simpler than in (x,y) coordinates!

QUESTION: Compute the area of a disk of radius R using a double integral

4) Double integrals in polar coordinates: more general regions

Consider now the more general region $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta)\}$



In this case, the double integral of f over the region D is written as follows in polar coordinates:

$$\boxed{\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

QUESTION: Find the area of the region given by the polar equation $r(\theta) = 4 + 3\cos\theta$ (this is the equation of the boundary enclosing the region)

QUESTION: Use double integrals in polar coordinates to compute the integral of the function $f(x,y) = x^2+y^2$ over the region enclosed by $(x-1)^2 + y^2 = 1$