Calculus II

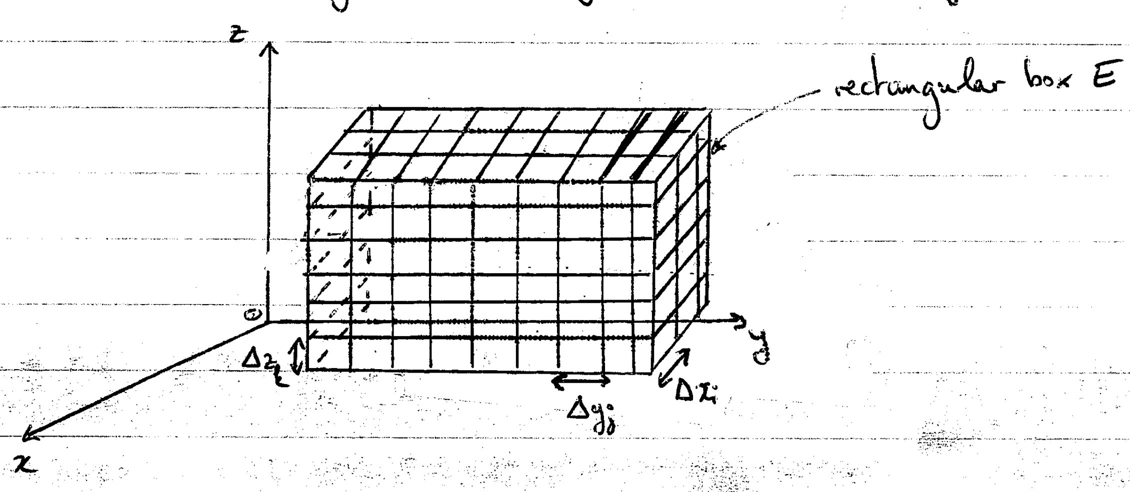
Lecture 20: Triple integrals

In the previous lectures, we have seen how to calculate the double integral of a function $f = f(x,y)$ of two variables.
In this lecture, we will see how we integrate a function $f = f(x,y,z)$ of three variables over a three-dimensional region E of space.
Because f is a function of three variables and because E is three-dimensional, the integral is a <u>triple integral</u>.

Note that the integral of a function of one variable represents an area; the integral of a function of two variables represents a volume. Since the graph of a function of three variables lives in a four-dimensional space, we do not have a simple geometric interpretation for triple integrals.

1) Mathematical construction of triple integrals

Triple integrals are constructed in a way very similar to double integrals. Suppose we want to integrate $f = f(x,y,z)$ over the rectangular box E below:



The idea, very similar to the double integral case, is to subdivide the region E in small rectangular boxes of width Δx_i, length Δy_j and height Δz_k. We then take a sample point (x_i^*, y_j^*, z_k^*) in each of the small boxes and the triple integral of f over the region E is approximated by:

$$\iiint_E f(x,y,z)\,dV \approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

$\Delta x_i \Delta y_j \Delta z_k$ is the volume of a small rectangular box:
$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$

In the limit of very small ΔV_{ijk}, the approximation above becomes more and more exact; so we have the following definition:

<u>Definition:</u> The triple integral of $f = f(x,y,z)$ over a region E is

$$\iiint_E f(x,y,z)\,dV = \lim_{\max(\Delta x_i, \Delta y_j, \Delta z_k)\to 0} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

This is how triple integrals are defined. In practice, as with double integrals, we will rarely use the definition to compute triple integrals. Instead, we will use iterated integrals.

2) Iterated integrals for triple integrals over rectangular boxes

Consider the rectangular box $B = [a,b] \times [c,d] \times [r,s]$. In order to compute $\iiint_B f(x,y,z)\,dV$, we use iterated integrals as follows:

- We first fix two variables, say y and z, and integrate $f(x,y,z)$ over all x in $[a,b]$ (at y and z constant)
- We then integrate the result over one of the previously fixed variables, say y, keeping z constant
- We finally integrate over z in $[r,s]$.

This is exactly what the next theorem says:

<u>Fubini's theorem for triple integrals</u>: If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

<u>Note</u>: As in the case of double integrals, <u>the order of integration can be exchanged.</u>
There are six possible orders. For instance, $\iiint_B f(x,y,z)\,dV$ can also be written as:

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

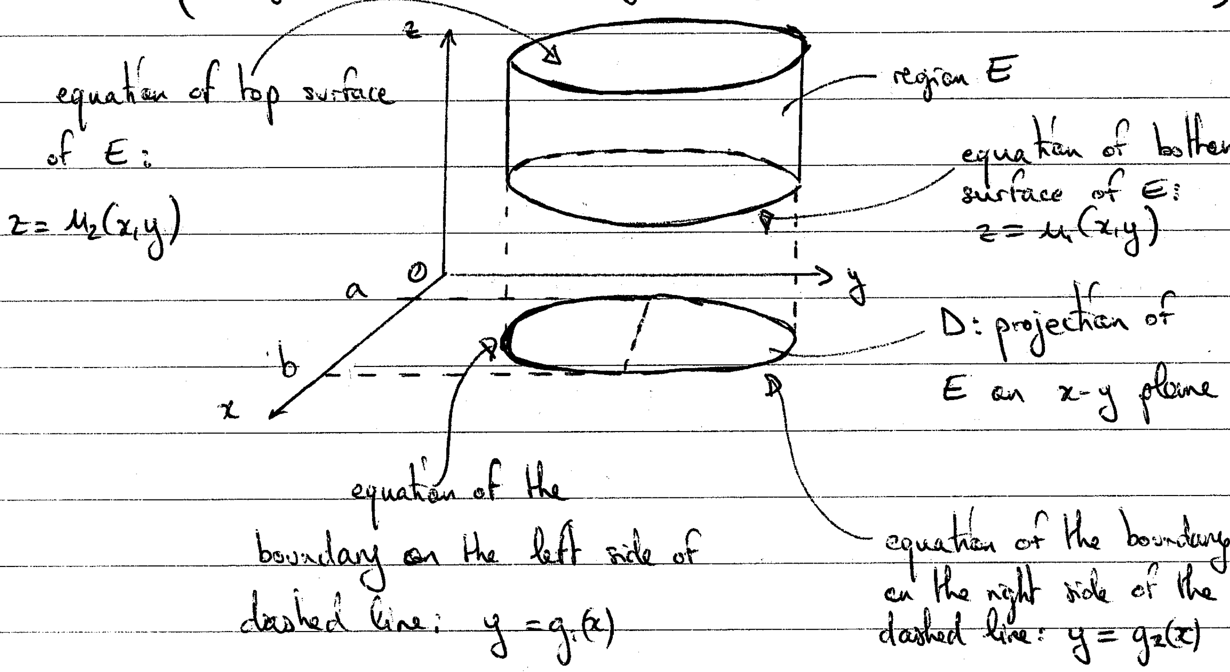
QUESTION Evaluate the triple integral $\int_0^1\int_0^{\pi/2}\int_0^1 yz\sin(xy)\,dx\,dy\,dz$

3) Iterated integrals for triple integrals over general regions

Iterated integrals also work for more general regions, but as in the case of double integrals, we now have to be careful with the limits of integration.

Illustration: * Say E is the region given by

$$E = \left\{ (x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y) \right\}$$



The triple integral of $f(x,y,z)$ over the region E is:

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Depending on how the region E is given to us, different permutations of the order of integration might be preferred. The formula given above is just one example. In general, the rule is to make sure the limits of integration are consistent with the mathematical description of the region E.

4) Properties of triple integrals

Triple integrals essentially have the same properties as seen in the case of double integrals: addition, multiplication by a scalar, estimates, etc.

In particular:

$$\boxed{\iiint_E 1 \, dV = V(E)}$$

where V is the volume of the region E

QUESTION: Use triple integrals to express the volume of the region between the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$

Evaluate the two inner integrals, and give the expression for the last integral