Calculus III

Lecture 21: Triple integrals in cylindrical coordinates

Just as in the case of double integrals, for some regions E and/or some functions $f = f(x,y,z)$ it will be more convenient to use different coordinates to evaluate triple integrals. In this class, we will see two alternate coordinate systems in three dimensions: cylindrical coordinates (this lecture) and spherical coordinates (next lecture).

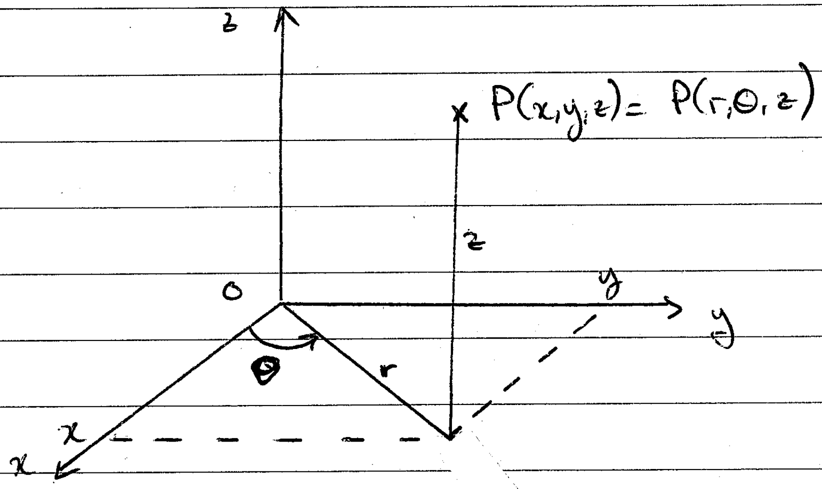
1) Cylindrical coordinates

Consider a point $P(x,y,z)$. The cylindrical coordinates for the point P, written $P(r,\theta,z)$ are defined as follows:

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

In other words, in order to go from Cartesian coordinates to cylindrical coordinates, one keeps the z value from the Cartesian coordinates and just replaces (x,y) by the polar coordinates (r,θ).

This is what is shown in the following figure:



Note that the transformation from Cartesian coordinates to cylindrical coordinates can be written as:

$$\boxed{r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

QUESTION: Plot the point with cylindrical coordinates $(\sqrt{2}, \frac{\pi}{4}, 1)$. Find its rectangular coordinates.

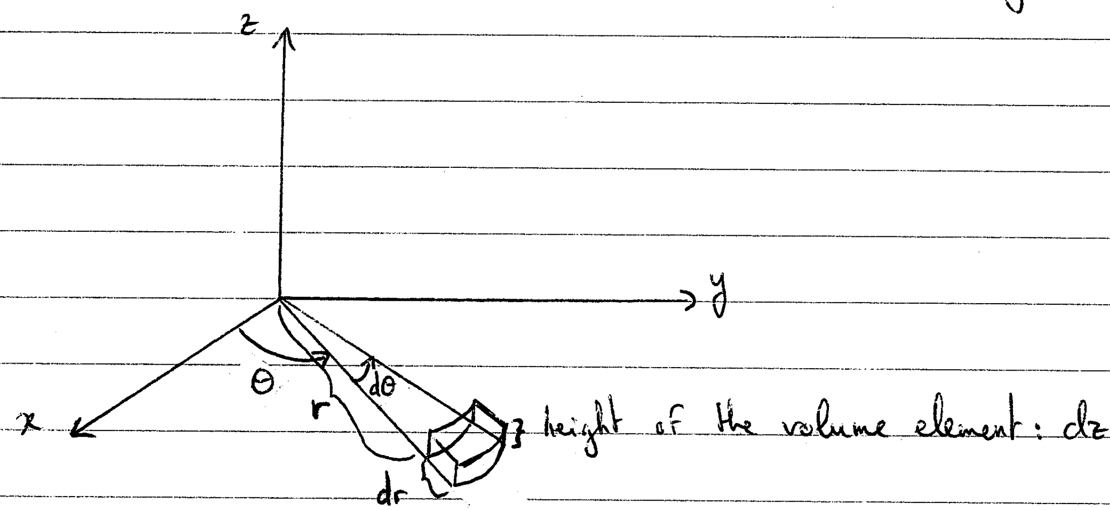
QUESTION: What is the equation of a circular cylinder with radius R, axis the z-axis and infinite length?

2) Triple integrals in cylindrical coordinates

- Element of volume in cylindrical coordinates

In rectangular coordinates, $dV = dx\,dy\,dz$. Let us calculate dV in cylindrical coordinates in order to be able to express triple integrals in these coordinates.

Let us draw an infinitesimal element of volume in cylindrical coordinates



The volume of the element is given by the formula: Area(base) x height

We know from the lecture on polar coordinates that the area of the base is $r\,dr\,d\theta$

The height of the element is dz.

We thus have:

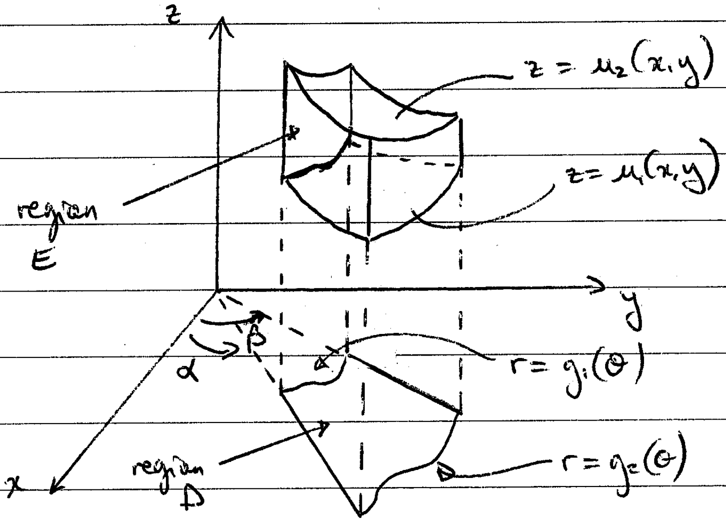
$$\boxed{dV = r\,dr\,d\theta\,dz}$$

- Triple integrals in cylindrical coordinates

Suppose we want to integrate a function $f = f(x,y,z)$ over a region E given by $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$

where D is given in polar coordinates by

$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ g_1(\theta) \leq r \leq g_2(\theta)\}$$



The integral of f is most conveniently evaluated in cylindrical coordinates, and we have:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

<u>QUESTION</u>: Use triple integrals in cylindrical coordinates to find the volume between the paraboloids $z = x^2+y^2$ and $z = 4-x^2-y^2$