Calculus III

Lecture 22: Triple integrals in spherical coordinates

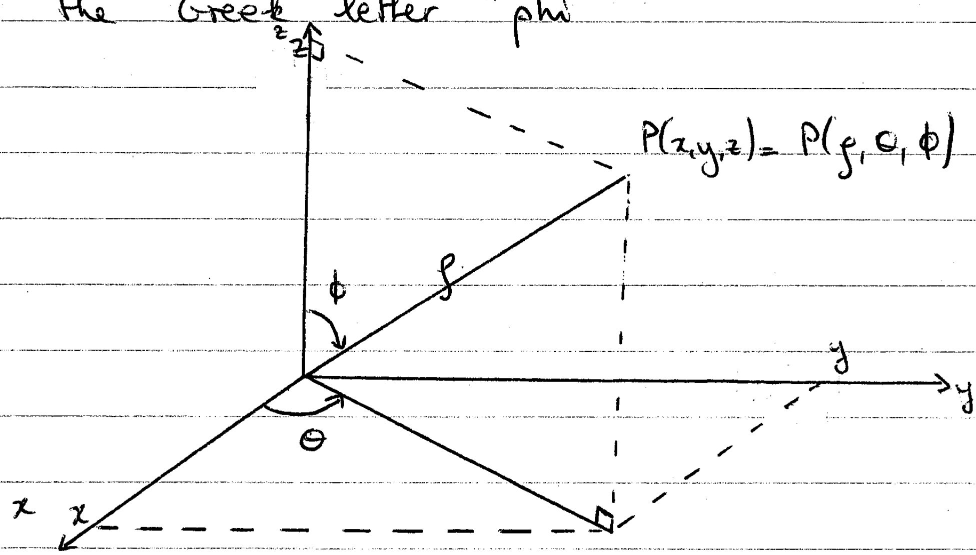
Another useful coordinate system in three dimensions is the spherical coordinate system. It usually simplifies the calculation of triple integrals over regions bounded by spheres or cones.

1) Spherical coordinates

Consider a point $P(x,y,z)$ in space. The spherical coordinates of the point P are (ρ, θ, ϕ) where:

- $\rho = |\vec{OP}|$; in other words, ρ is the distance from the origin to P
- θ is the same angle as in cylindrical coordinates
- ϕ is the angle between the positive z-axis and the line segment OP

Note: ρ is the Greek letter "rho", θ is the Greek letter "theta", and ϕ is the Greek letter "phi"



Relationship between spherical and cylindrical co-ordinates

We can use a little trigonometry and the previous figure to find that:

$$z = \rho\cos\phi \qquad r = \rho\sin\phi \qquad \theta = \theta$$

These equations give the transformation $(\rho, \theta, \phi) \rightarrow (r, \theta, z)$ from spherical coordinates to cylindrical coordinates.

Relationship between spherical and rectangular coordinates

We know from previous lectures the relationship between the polar coordinates (r, θ) and the rectangular coordinates (x, y):

$$x = r\cos\theta \qquad y = r\sin\theta$$

Using the expression for r above gives us the transformation from spherical coordinates to rectangular coordinates:

$$\boxed{x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi}$$

The inverse transformation, from rectangular coordinates to spherical coordinates is given by

$$\rho = \sqrt{x^2+y^2+z^2} \qquad \tan\theta = \frac{y}{x} \qquad \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

QUESTION: Consider the point $P\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$. What are the rectangular coordinates of P?

QUESTION: Describe the surface with the following equation in spherical coordinates:

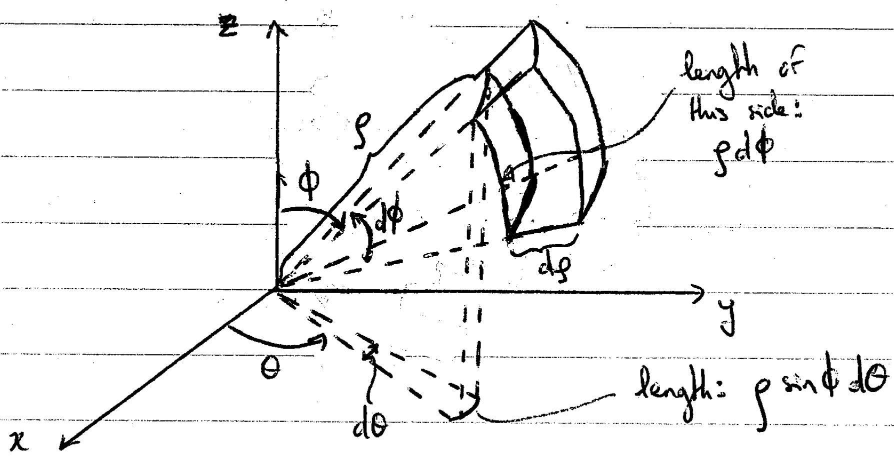
1) $\rho = 1$

2) $\phi = \frac{\pi}{4}$

2) Triple integrals in spherical coordinates

- Element of volume in spherical coordinates

Let us look at an element of volume in spherical coordinates:



Note: the size of the element of volume has been magnified; you should imagine a small wedge which almost looks like a rectangular box

In the limit in which the wedge is very small, it looks like a rectangular box, and its volume is found by multiplying the lengths of the sides of the box.

As shown on the figure,

- the side in the radial direction has length $d\rho$
- the side in the direction of the ϕ arrow has length $\rho d\phi$
- the side in the direction of the θ arrow has length $\rho \sin\phi d\theta$

Therefore, a small volume element in spherical coordinates has the volume:

$$dV = d\rho \, \rho d\phi \, \rho \sin\phi d\theta$$

$$\Leftrightarrow \boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

- <u>Triple integrals in spherical coordinates</u>

Consider the following three-dimensional region E in spherical coordinates:

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

We can use our formula for dV to obtain the following formula for the triple integral in spherical coordinates over the region E:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

- **Extension to slightly more general regions**

Consider a region E which can be written as

$$E = \{(\rho,\theta,\phi) \mid \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d,\ g_1(\theta,\phi) \leq \rho \leq g_2(\theta,\phi)\}$$

then the triple integral can be written as

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

QUESTION: Evaluate $\iiint_B \sqrt{x^2+y^2+z^2}\,dV$ where B is the ball given by

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 3\}$$