Calculus III

Lecture 23: Vector Fields

With this lecture, we start the last part of the course, in which we study vector calculus. As its name suggests, vector calculus is the calculus of vector fields. In this first lecture on vector calculus, we thus need to define what we mean by vector fields.
We will start with the mathematical definition, then study simple examples and use real life situations to explain why they are important.

1) Definitions

- Definition in 2 dimensions:

Let D be a set in $\mathbb{R}^2$ (i.e a plane region). A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\vec{F}(x,y)$.

- Definition in 3 dimensions:

Let E be a set in $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x,y,z) in E a three-dimensional vector $\vec{F}(x,y,z)$.

*Thus far, we had seen functions that assigned a scalar value to any point (x,y) in $\mathbb{R}^2$ (or any point (x,y,z) in $\mathbb{R}^3$). Vector fields are functions that do not assign scalar values but <u>vectors</u> to any point in $\mathbb{R}^2$ (or in $\mathbb{R}^3$).

* Since $\vec{F}(x,y)$ is a two-dimensional vector, we can write it in terms of its component functions P and Q as follows:

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P(x,y), Q(x,y) \rangle$$

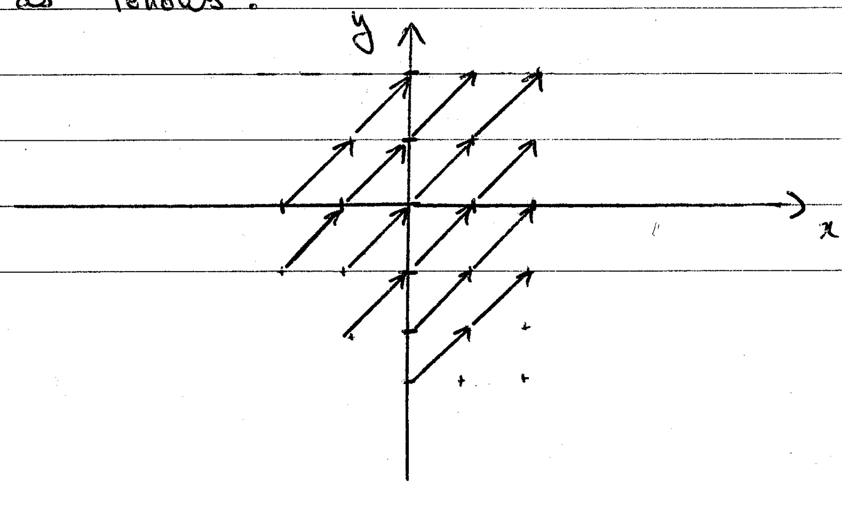
Here, P and Q are just functions of two variables.

* Likewise $\vec{F}(x,y,z)$ is a three-dimensional vector, which we can write in terms of its component functions P, Q, and R:

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

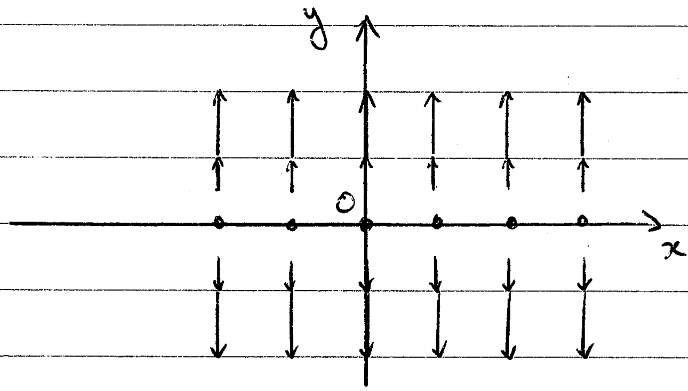
2) <u>Examples of vector fields</u>

- Let us first consider the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \vec{i} + \vec{j}$. We see that whatever the point (x,y), $\vec{F}$ always assigns the vector $\langle 1,1 \rangle$. Thus $\vec{F}$ looks as follows:

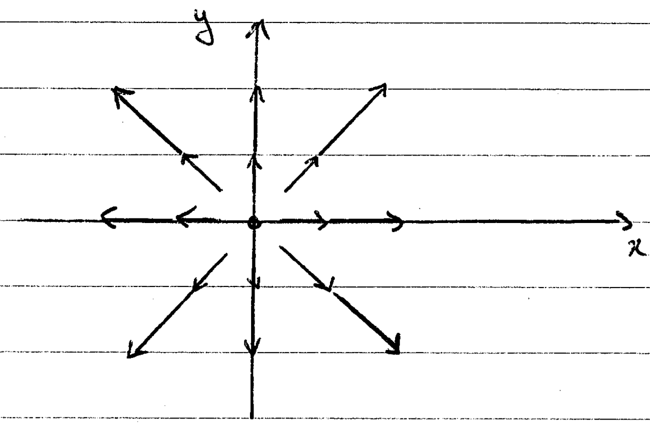


Note that on the previous figure, we have to imagine vectors starting at each and every point (x,y) in the plane. The reason we do not draw all the vectors is that they would all run over each other, and the figure would be hard to read.

- Let us now try and sketch another vector field: $\vec{F}(x,y) = y\vec{j}$
It looks as follows:



- Let us look at a last example: $\vec{F}(x,y) = x\vec{i} + y\vec{j}$
For this example, at each point $P(x,y)$ the vector $\vec{F}(x,y) = \langle x, y \rangle$ is equal to the position vector $\vec{OP}$. Thus, the vector field looks as follows:



<u>Note:</u> In practice, we will rarely plot vector fields; computers are quite good at doing this. However, it is good to be able to guess and recognize the general patterns of given vector fields

QUESTION: Sketch some of the vectors of the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$

3) Relevance and importance of vector fields

Vector fields are very convenient objects to represent patterns for physical phenomena we experience in our daily life. For example, if you answered the previous question in the right way, the pattern you found should be very close to the pattern the velocity of coffee in you cup follows when you stir your coffee with a spoon.

In a similar way, vector fields are commonly used to represent ocean currents to help sailors.
Vector fields are also used to represent wind patterns, the direction of the pull of gravity everywhere on Earth, etc.

4) A vector field we have known for a while: the gradient vector field

If f is a scalar function of two variables, then recall from lecture 15 that the gradient of f, $\vec{\nabla} f$, is defined by:

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$

We see that $\vec{\nabla} f$ is a vector field on $\mathbb{R}^2$, its component functions being $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\vec{\nabla} f$ is therefore called a gradient vector field.

Likewise, if $f = f(x,y,z)$ is a function of three variables, its gradient

$$\vec{\nabla} f = \frac{\partial f}{\partial x}(x,y,z)\,\vec{i} + \frac{\partial f}{\partial y}(x,y,z)\,\vec{j} + \frac{\partial f}{\partial z}(x,y,z)\,\vec{k}$$

is a vector field on $\mathbb{R}^3$, called a gradient vector field.

QUESTION: Find the gradient vector field of $f(x,y) = -\frac{x^2+y^2}{2}$ and sketch some of its vectors