Calculus III

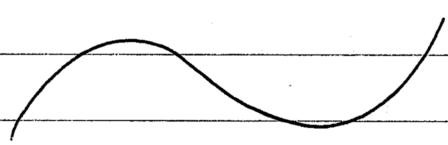
Lecture 26: Green's Theorem

Green's theorem, the topic of today's lecture, can be seen as the counterpart of the Fundamental Theorem Calculus for double integrals. We will first give its exact statement, and then see how to use it in practice in future lectures.

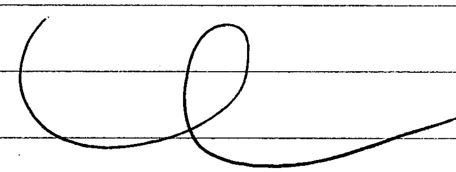
I] Green's Theorem for a simply connected region

1) Simple closed curve

A simple curve is a curve that does not intersect itself anywhere between its endpoints.

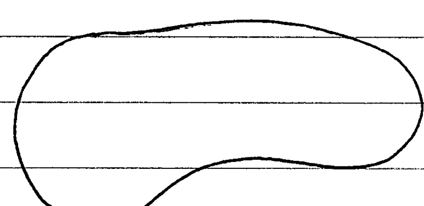


simple curve

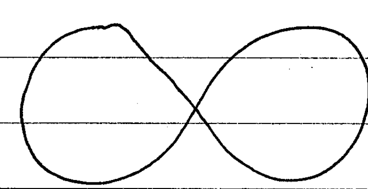


not simple curve

A simple closed curve is a simple curve whose initial and terminal point coincide.



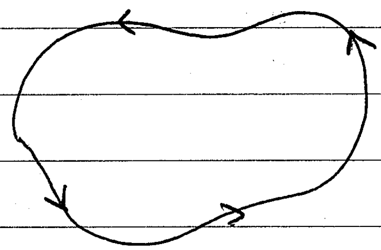
simple closed curve



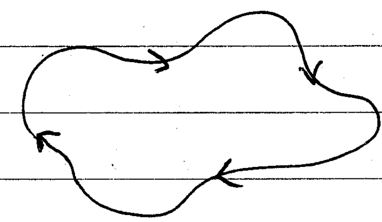
closed, not simple curve

2) Orientation of a curve

The orientation of a closed curve is the direction in which the curve is traversed. By convention, the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.



Positive orientation



Negative orientation

3) Green's Theorem

Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then

$$\boxed{\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Note 1: We will not prove this theorem in class, but you can find a proof in the textbook

Note 2: By taking $Q = 0$, Green's Theorem implies

$$\int_C P(x,y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

Likewise, by taking $P = 0$, Green's Theorem implies

$$\int_C Q(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

With these formulas, we better understand why Green's Theorem can be seen as a counterpart of the Fundamental Theorem of Calculus for double integrals

Note 3: ⚠ When using Green's Theorem, always make sure that the curve is closed

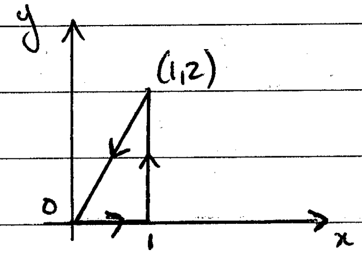
Note 4: To remind the reader that the integral is along a simple closed curve, one often uses the notation:

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

To remind the reader of the positive orientation of the curve, one can use the following notation:

$$\ointctrclockwise P(x,y)\,dx + Q(x,y)\,dy$$

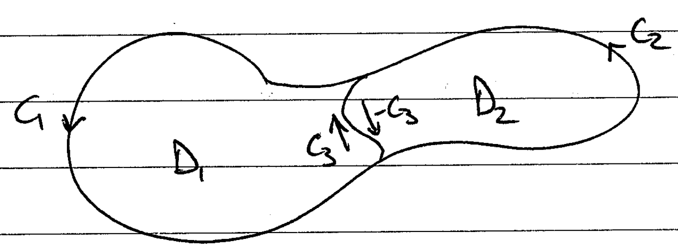
QUESTION: Use Green's Theorem to evaluate $\int_C x^2y\,dx + x\,dy$ along the following curve C:



II| Extension of Green's Theorem to more general regions

1) Finite union of simple regions

Imagine D is the union of two non overlapping regions D_1 and D_2:



$D = D_1 \cup D_2$

The boundary of D_1 is then the positively oriented simple closed curve $C_1 \cup C_3$, and we can write

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

using Green's theorem.

Likewise, the boundary of D_2 is $C_2 \cup (-C_3)$, and according to

Green's theorem we have:

$$\int_{C_2 \cup (-C_3)} P(x,y)\,dx + Q(x,y)\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

We can add the two equalities. The line integrals along C_3 and $-C_3$ cancel, so we find:

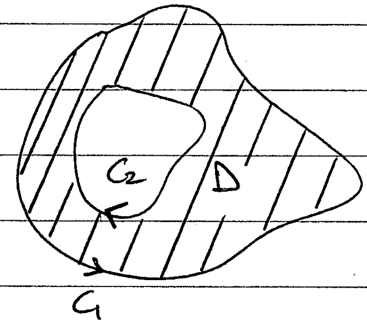
$$\int_{C_1 \cup C_2} P(x,y)\,dx + Q(x,y)\,dy = \iint_{D_1 \cup D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

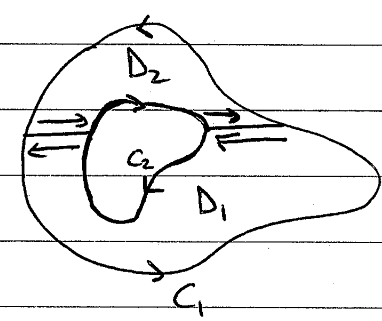
$$\iff \int_{C} P(x,y)\,dx + Q(x,y)\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

So Green's theorem holds in this case too, because the boundary of the region $D = D_1 \cup D_2$ is $C = C_1 \cup C_2$.

2) Regions with holes

Consider the region D as follows:





The boundary $C = C_1 \cup C_2$ of the region D is made of two simple closed curve C_1 and C_2. The idea is to calculate the integral along C

by decomposing D in two regions D_1 and D_2 as shown in the figure on the right-hand side.

We can write $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

The boundary of D_1 is a positively oriented simple closed curve. So is the boundary of D_2. We can therefore use Green's Theorem as follows:

$$\iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\substack{\text{Boundary} \\ \text{of } D_1}} P\,dx + Q\,dy + \oint_{\substack{\text{Boundary} \\ \text{of } D_2}} P\,dx + Q\,dy$$

Now, note that the line integrals along the common boundary lines are in opposite direction, so they cancel. The right-hand side can therefore be rewritten as:

$$\oint_{\substack{\text{Boundary} \\ \text{of } D_1}} P\,dx + Q\,dy + \oint_{\substack{\text{Boundary} \\ \text{of } D_2}} P\,dx + Q\,dy = \oint_{C_1} P(x,y)\,dx + Q(x,y)\,dy + \oint_{C_2} P(x,y)\,dx + Q(x,y)\,dy$$

In other words, we proved that:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1} P(x,y)\,dx + Q(x,y)\,dy + \oint_{C_2} P(x,y)\,dx + Q(x,y)\,dy$$

This is an extension of Green's Theorem to regions with a hole. Observe that the integral around the outer boundary is taken counterclockwise and the integral around the hole is taken clockwise.

QUESTION: Evaluate the integral $\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}$

where C is any piecewise smooth simple closed curve oriented counterclockwise such that:

1st case: C does not enclose the origin

2nd case: C encloses the origin

IV Finding areas using Green's Theorem

Sometimes it can be easier to evaluate a line integral than a double integral, and that is where Green's Theorem can help to calculate the area of a region.

Remember from Lecture 18 that the area of a region D of space is

$$A(D) = \iint_D 1\,dA$$

By choosing $P(x,y)$ and $Q(x,y)$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, we can use Green's Theorem to write:

$$A(D) = \iint_D 1\,dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

For example, with $P(x,y) = 0$ and $Q(x,y) = x$, we have

$$A(D) = \oint_C x\,dy$$

With $P(x,y) = -y$ and $Q(x,y) = 0$, we have

$$A(D) = -\oint y\,dx$$

With $P(x,y) = -\frac{y}{2}$ and $Q(x,y) = \frac{x}{2}$, we have

$$A(D) = \frac{1}{2}\oint x\,dy - y\,dx$$

where C is the simple closed curve enclosing the region D.

Each formula has advantages depending on the situation. Even though the third formula looks more complicated, it often leads to simple integrations.

QUESTION: Use a line integral to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$