Calculus III

Lecture 30: The divergence theorem

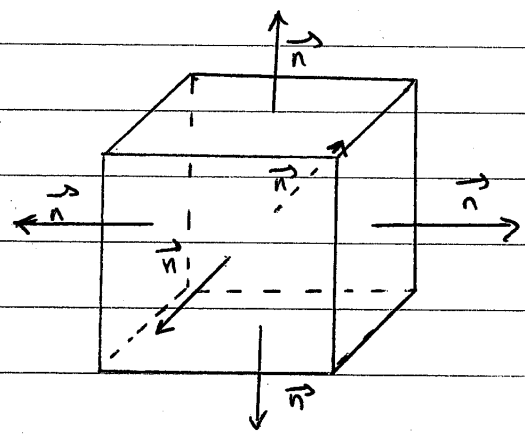
In this lecture, we learn how the surface integrals of vector fields we saw in the previous lecture can be turned into volume integrals over the volume enclosed by the surface of interest. The transformation involves the divergence of the vector field, and is called the divergence theorem.

1) The divergence theorem

Theorem: Let E be a simple solid region (e.g. a region bounded by a rectangular box or bounded by an ellipsoid) and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then,

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV}$$

• Illustration:



The surface S is the union of all the sides of the cube, they all have positive orientation, given by the vectors $\vec{n}$

The region E is the volume in the cube

The theorem says that the total flux of a vector field through the surface S is equal to the integral of the divergence of the vector field over the region E.

• **Note**: We will not prove the theorem in these notes, but you can find a clear and instructive proof in the textbook.

2) Applying the divergence theorem

* Sometimes it is easier to find the flux across a closed surface by using the Divergence Theorem than by using the direct surface integral we learned about in the last lecture. This is what we show here, in our first application of the Divergence Theorem:

Use the Divergence Theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = z\,\vec{k}$ across the sphere $x^2+y^2+z^2=a^2$

Answer: Let S denote the outward-oriented spherical surface and E the region that S encloses. The divergence of $\vec{F}$ is:

$$\operatorname{div}\vec{F} = \frac{\partial z}{\partial z} = 1$$

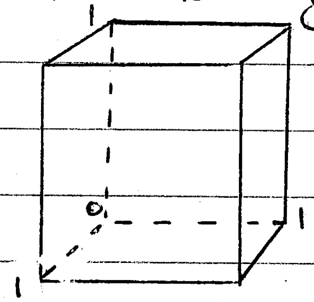
Hence, using the Divergence Theorem, we find

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 1\,dV = V(E) = \frac{4\pi a^3}{3}$$

You can now see that it is much easier to answer this question with the Divergence Theorem than with the method we used for the same question in the previous lecture.

* The Divergence Theorem is usually the method of choice for finding the flux across closed piecewise smooth surfaces with multiple sections since it eliminates the need for a separate integral evaluation over each section. Here is an example illustrating this:

Use the Divergence Theorem to find the outward flux of the vector field $\vec{F} = 2x\vec{i} + 3y\vec{j} + z^2\vec{k}$ across the unit cube.



If we calculated the flux with the direct method, we would need to set up and calculate six different surface integrals, for each face of the cube. Instead, with the divergence theorem, we will only need to calculate 1 volume integral.

$$\text{div}\,\vec{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 2 + 3 + 2z = 5 + 2z$$

Let S be the surface of the cube, and E the volume it encloses. According to the divergence theorem, we have:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 (5+2z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[5z + z^2\right]_0^1 dy\,dx = \int_0^1\int_0^1 6\,dy\,dx = 6$$

3) A cautionary tale for the divergence theorem

When you want to use the Divergence Theorem, you should make sure that the vector field $\vec{F}$ is well-defined in the region E where you are using the theorem, and that the component functions of $\vec{F}$ indeed have continuous partial derivatives, as stated in the theorem. Otherwise, you may get complete rubbish. Here is an example of this:

Consider the vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \dfrac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \dfrac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k}$

Show that the outward flux of $\vec{F}$ across any closed orientable surface S surrounding the origin is 4π.

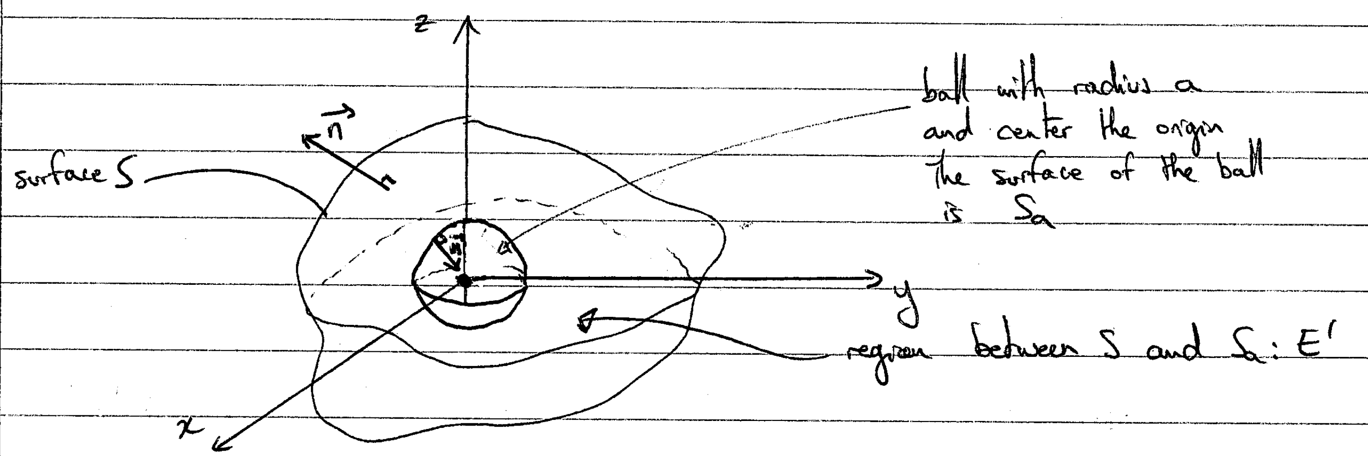
For $(x,y,z) \neq (0,0,0)$,

$$\begin{aligned} \operatorname{div}\vec{F} = {} & (x^2+y^2+z^2)^{-3/2} - x\frac{3}{2}(2x)(x^2+y^2+z^2)^{-5/2} \\ & + (x^2+y^2+z^2)^{-3/2} - y\frac{3}{2}(2y)(x^2+y^2+z^2)^{-5/2} \\ & + (x^2+y^2+z^2)^{-3/2} - z\frac{3}{2}(2z)(x^2+y^2+z^2)^{-5/2} \end{aligned}$$

$$\operatorname{div}\vec{F} = 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) = 0$$

Since $\operatorname{div}\vec{F} = 0$ for $(x,y,z) \neq (0,0,0)$, a careless application of the Divergence Theorem might lead us to think that the flux of $\vec{F}$ across S is zero. This is not true. Of course, the reason is that we cannot use the Divergence Theorem as such since the region E contains the point $(0,0,0)$ (the origin) where neither $\vec{F}$ nor the derivatives of its component functions are defined.

The idea to compute the flux of $\vec{F}$ across S is to define a new region, called E', which the the region E minus the ball with center the origin and radius a:



E' does not include the origin, so we can use the Divergence Theorem in E':

$$\iiint_{E'} \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_a} \vec{F} \cdot d\vec{S}$$

Since in E' $\operatorname{div} \vec{F} = 0$, the left-hand side is zero:

$$0 = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_a} \vec{F} \cdot d\vec{S} \iff \iint_S \vec{F} \cdot d\vec{S} = -\iint_{S_a} \vec{F} \cdot d\vec{S}$$

So the integral we are after, $\iint_S \vec{F} \cdot d\vec{S}$, is just the negative of the inward flux of $\vec{F}$ across S_a. The latter can be calculated as follows:

$$\iint_{S_a} \vec{F} \cdot d\vec{S} = \iint_{S_a} \vec{F} \cdot \vec{n}\, dS = -\iint_{S_a} \vec{F} \cdot \frac{\vec{r}}{|\vec{r}|}\, dS$$ since $-\frac{\vec{r}}{|\vec{r}|}$ is the inward unit normal vector for the sphere S_a

$$= -\iint_{S_a} \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|}\, dS \qquad \text{since } \vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$$

$$= -\iint_{S_a} \frac{|\vec{r}|^2}{|\vec{r}|^4}\, dS = -\iint_{S_a} \frac{1}{|\vec{r}|^2}\, dS = -\frac{1}{a^2}\iint_{S_a} dS \qquad (\text{since } |\vec{r}| = a \text{ on } S_a)$$

$$= -\frac{1}{a^2} A(S) = -\frac{1}{a^2} \times \underbrace{4\pi a^2}_{\text{area of sphere of radius } a}$$

$$= -4\pi$$

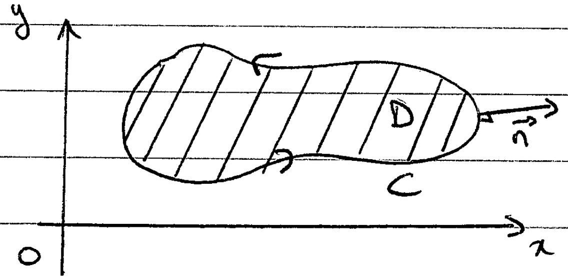
Thus $\iint_{S_a} \vec{F} \cdot d\vec{S} = -4\pi$, so that $\underline{\iint_{S} \vec{F} \cdot dS = 4\pi}$

This result, known as Gauss's law for inverse-square fields, is independent of the shape of the closed surface S. It is a key result in many fields of physics.
The important point to remember is that we did not find 0, even though $\operatorname{div} \vec{F} = 0$ everywhere where $\vec{F}$ is defined. This is because one can only apply the Divergence Theorem to regions E where $\vec{F}$ is everywhere defined, and the component functions have continuous partial derivatives.

4) Reflecting back on the vector form of Green's theorem using the divergence

Remember the vector form of Green's theorem using the divergence:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}\,dA$$



The equality can be interpreted as the <u>2D version of the Divergence Theorem</u>: the flux of $\vec{F}$ across the boundary C is equal to the integral of $\operatorname{div}\vec{F}$ over the region D.