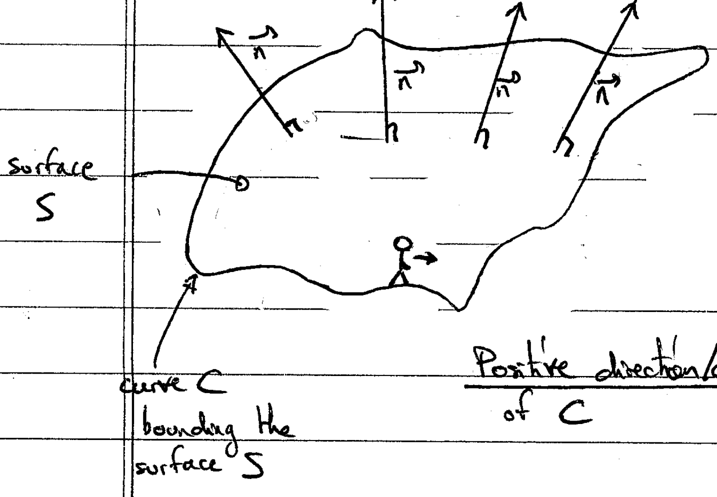
Calculus III

Lecture 31: Stokes' Theorem

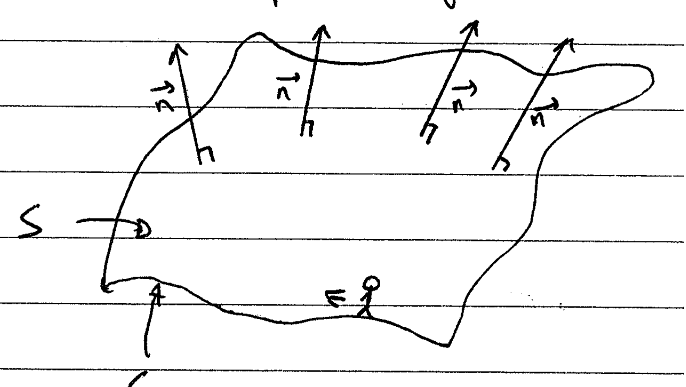
In this lecture, we learn how line integrals of vector fields along closed curves can be turned into surface integrals. The result is called Stokes' theorem, and can be seen as a generalization to three dimensions of the vector form of Green's theorem with the curl.

1) Relative orientation of curves and surfaces

Consider an oriented surface S in three dimensional space, bounded by a simple closed parametric curve C. There are two possible relationships between the orientations of S and C, which can be described as follows. Imagine a person walking along the curve C with his or her head in the direction of the orientation of S. The person is said to be walking in the positive direction of C relative to the orientation of S if the surface is on the person's left.
The person is said to be walking in the negative direction of C relative to the orientation of S if the surface is on the person's right.



Positive direction/orientation of C



Negative direction/orientation of C

Here is another way to understand positive orientation vs negative orientation: if the fingers of your right hand are cupped in the positive direction of C, then your thumb points (roughly) in the direction of the orientation of S.

2) Stokes' Theorem

Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}}$$

Interpretation: Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\vec{F}$ is equal to the surface integral of the normal component of the curl of $\vec{F}$

Note: The proof of Stokes' theorem is beyond the scope of this class. Instead, we will focus on applications of the theorem.

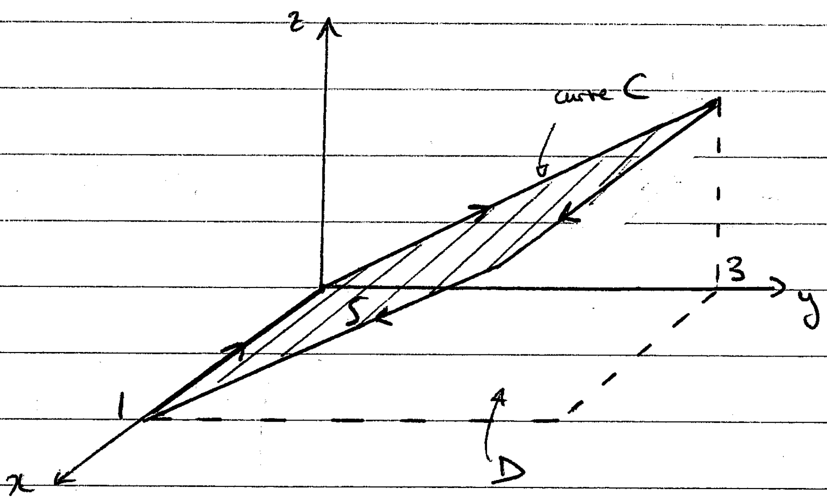
3) Application of Stokes' Theorem

* Stokes' Theorem can be very helpful when we want to calculate the line integral of a vector field along a piecewise smooth curve with multiple sections. Instead of subdividing the integral into multiple line integrals along each section, we can use Stokes' theorem to compute only one surface integral. Here is an illustration:

Find the work performed by the vector field

$$\vec{F}(x,y,z) = x^2\vec{i} + 4xy^3\vec{j} + y^2x\vec{k}$$

on a particle that traverses the rectangle C in the plane $z=y$ shown in the figure below:



The work performed by the field is $W = \oint \vec{F} \cdot d\vec{r}$

To calculate this directly, we would have to consider each side of the rectangle independently. Instead, we use Stokes' Theorem and write:

$$W = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

Here we take S oriented downward, so the orientation of C is positive. Hence, we write:

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} \underset{\substack{\text{downward}\\ \text{orientation}\\ \text{of } S}}{=} -\iint_D \text{curl}\,\vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\,dx\,dy$$

- The surface S is given by the equation $z = y$, which can be written as the following vector equation: $\vec{r} = x\vec{i} + y\vec{j} + y\vec{k}$

$\vec{r}_x = \vec{i} \qquad \vec{r}_y = \vec{j} + \vec{k}$

$$\vec{r}_x\times\vec{r}_y = \vec{i}\times\vec{j} + \vec{i}\times\vec{k} = \vec{k} - \vec{j}.$$

- $$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix} = 2xy\,\vec{i} - y^2\vec{j} + 4y^3\vec{k}$$

Thus,

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = -\int_0^3\int_0^1 (2xy\vec{i} - y^2\vec{j} + 4y^3\vec{k})\cdot(\vec{k} - \vec{j})\,dx\,dy$$

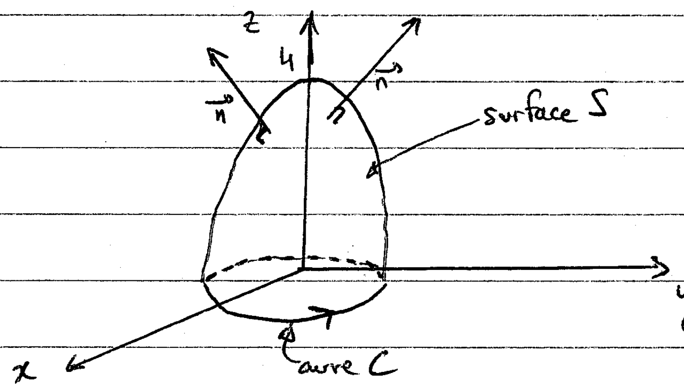
$$= -\int_0^3\int_0^1 (4y^3 + y^2)\,dx\,dy = -\int_0^3 (4y^3+y^2)\Big[x\Big]_0^1\,dy$$

$$= -\left[y^4 + \frac{y^3}{3}\right]_0^3 = -(81+9)$$

$$= -90$$

* The opposite situation may also occur: sometimes a line integral is easier to evaluate than a surface integral, as in the following example:

Compute the integral $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = 2z\vec{i} + 3x\vec{j} + 5y\vec{k}$ and where S is the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation.



Using Stokes' Theorem, we can write $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

The curve C is in the plane $z = 0$ and has the equation $x^2 + y^2 = 4$. Parametric equations for this curve are:

$$x = 2\cos t \qquad y = 2\sin t \qquad z = 0 \qquad 0 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = 2\cos t \qquad \frac{dz}{dt} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C 2z\,dx + 3x\,dy + 5y\,dz$$

$$= \int_0^{2\pi} 2\underbrace{z(t)}_{0} \frac{dx}{dt} dt + \int_0^{2\pi} 3x(t) \frac{dy}{dt} dt + \int_0^{2\pi} 5y(t) \underbrace{\frac{dz}{dt}}_{0} dt$$

$$= 3\int_0^{2\pi} 2\cos t \times 2\cos t\,dt = 12\int_0^{2\pi} \cos^2 t\,dt = 12\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= 12\left[\frac{t}{2} + \frac{\sin(2t)}{4}\right]_0^{2\pi} = 12\pi$$

<u>Note:</u> In the previous example, we computed a surface integral simply by knowing the value of $\vec{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}$$

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4) <u>Reflecting back on the vector form of Green's Theorem using curl</u>

In lecture 27, we obtained the following vector form of Green's Theorem using curl:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \vec{k} \, dA$$

We can now see that this is just a particular case of Stokes' Theorem when the surface S is entirely in the xy plane, so that the normal vector is $\vec{k}$: