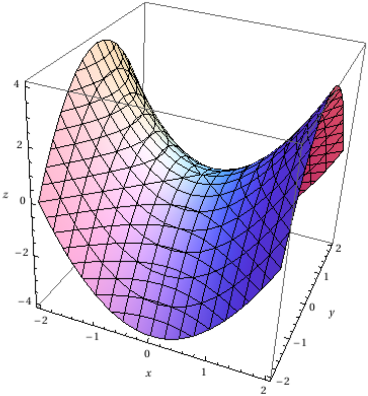
Calculus III
Lecture 6: Cylinders and quadric surfaces



(Section 10.6)

Two parts in this lecture

1. Cylinders

2. Quadric surfaces

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1. Cylinders

2. Quadric surfaces

Cylinders

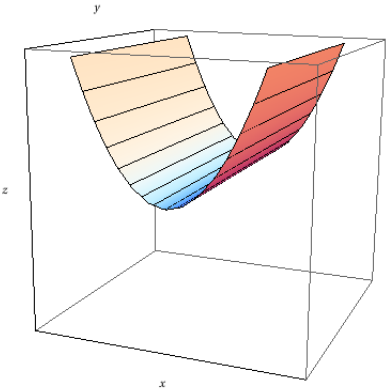
Definition: A cylinder is a **surface** that consists of all lines that are parallel to a given line and pass through a given plane curve.

Example: The surface of equation $z = x^2$

- y does not enter in the equation $\rightarrow$ let's look at the trace of the surface $z = x^2$ on the plane $y = k$
- For a given $y = k$, a point P(x,y,z) belongs to the surface if $z = x^2$. This means that the intersection of the surface with the plane $y = k$ is the **parabola** $z = x^2$
- We obtain the full surface by *assembling* together the infinitely many parabolas traced in each plane

CYLINDERS

The surface of equation $z = x^2$: graphical representation



The lines shown in the plot are the lines mentioned in the definition. They are all parallel to the y axis and pass through the plane curves $z = x^2$

Cylinders

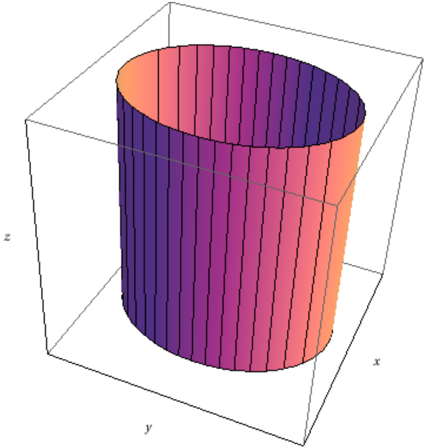
Example 2: The surface of equation $\frac{x^2}{4} + \frac{y^2}{64} = 1$

Cylinders

Example 2: The surface of equation $\frac{x^2}{4} + \frac{y^2}{64} = 1$

- z does not enter in the equation
- In each plane $z = k$, $\frac{x^2}{4} + \frac{y^2}{64} = 1$ describes an ellipse
- The surface given by $\frac{x^2}{4} + \frac{y^2}{64} = 1$ is a cylinder whose axis is the z axis and whose cross-section is an ellipse: it is called an **elliptic cylinder**

The surface of equation $\frac{x^2}{4} + \frac{y^2}{64} = 1$: Graphical illustration of the elliptic cylinder



Two parts in this lecture

1. Cylinders

2. Quadric surfaces

Quadric surfaces

A **quadric surface** is a surface given by the general **second-degree** equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, \ldots, J$ are all constants

It turns out that by translations and rotations of the surface, it can always be brought in the following 2 standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \tag{1}$$

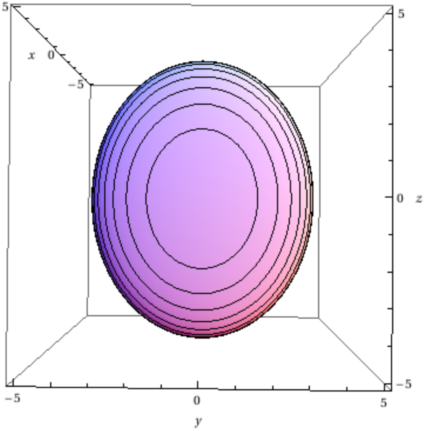
OR

$$Ax^2 + By^2 + Iz = 0 \tag{2}$$

The purpose of the remainder of the lecture is to learn about all the generic shapes that are determined by equations of the form (1) or (2)

Ellipsoid

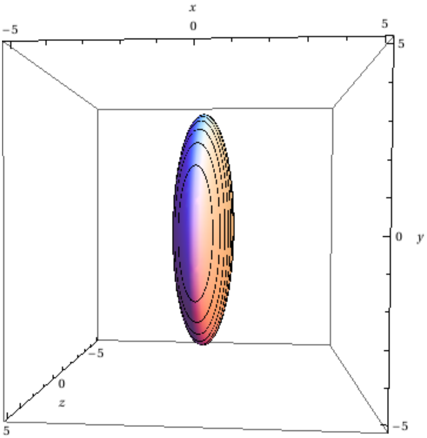
Example: Surface given by the equation $x^2 + \frac{y^2}{16} + \frac{z^2}{25} = 1$



- For each $x = k$ fixed, the equation of the surface is $\frac{y^2}{16} + \frac{z^2}{25} = 1 - k^2$
- $\frac{y^2}{16} + \frac{z^2}{25} = 1 - k^2$ is the equation of an *ellipse* for $-1 < k < 1$
- Hence the trace of the surface on the planes $x = k$ are ellipses (see figure)

Ellipsoid

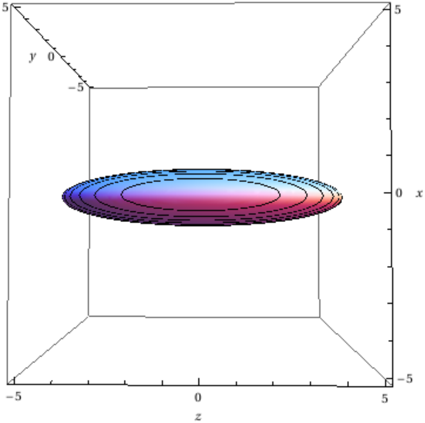
Example: Surface given by the equation $x^2 + \frac{y^2}{16} + \frac{z^2}{25} = 1$



- For each $z = k$ fixed, the equation of the surface is $x^2 + \frac{y^2}{16} = 1 - \frac{k^2}{25}$
- $x^2 + \frac{y^2}{16} = 1 - \frac{k^2}{25}$ is the equation of an *ellipse* for $-5 < k < 5$
- Hence the trace of the surface on the planes $z = k$ are ellipses (see figure)

Ellipsoid

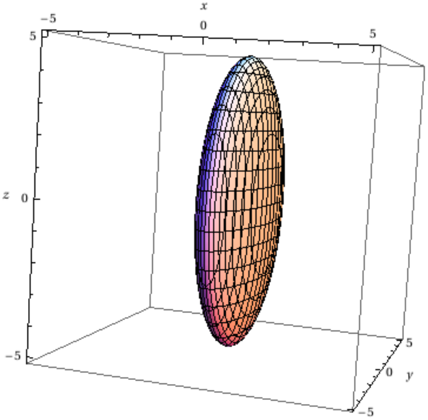
Example: Surface given by the equation $x^2 + \frac{y^2}{16} + \frac{z^2}{25} = 1$



- For each $y = k$ fixed, the equation of the surface is $x^2 + \frac{z^2}{25} = 1 - \frac{k^2}{16}$
- $x^2 + \frac{z^2}{25} = 1 - \frac{k^2}{16}$ is the equation of an *ellipse* for $-4 < k < 4$
- Hence the trace of the surface on the planes $y = k$ are ellipses (see figure)

ELLIPSOID

Surface given by the equation $x^2 + \frac{y^2}{16} + \frac{z^2}{25} = 1$: Graphical illustration



Elliptic paraboloid

Example: Surface given by the equation $3x^2 + 5y^2 = z$

Elliptic paraboloid

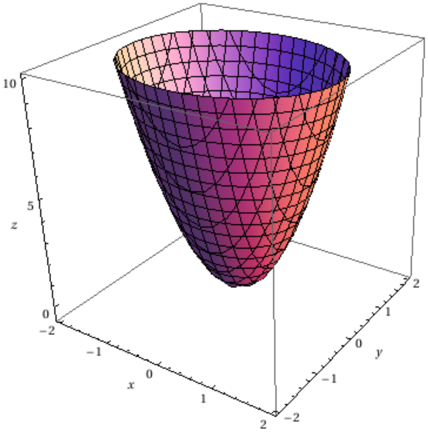
Example: Surface given by the equation $3x^2 + 5y^2 = z$

- In each plane $x = k$, the surface has a trace given by the equation $z = 5y^2 + 3k^2$: this is the equation of a **parabola**
- In each plane $y = k$, the surface has a trace given by $z = 3x^2 + 5k^2$: this is also the equation of a **parabola**
- In each plane $z = k$, the surface has a trace given by $3x^2 + 5y^2 = k$, which can be rewritten as $\frac{x^2}{5} + \frac{y^2}{3} = \frac{k}{15}$: this is the equation of an **ellipse** for $k \geq 0$

The surface is an **elliptic paraboloid**

Elliptic paraboloid

Surface given by the equation $3x^2 + 5y^2 = z$: Graphical illustration



Hyperboloid

Example: Surface given by the equation $\frac{x^2}{2} + y^2 - \frac{z^2}{3} = 1$

HYPERBOLOID

Example: Surface given by the equation $\frac{x^2}{2} + y^2 - \frac{z^2}{3} = 1$

- In each plane $x = k$, the surface has a trace given by the equation $y^2 - \frac{z^2}{3} = 1 - \frac{k^2}{2}$: this is the equation of a **hyperbola**

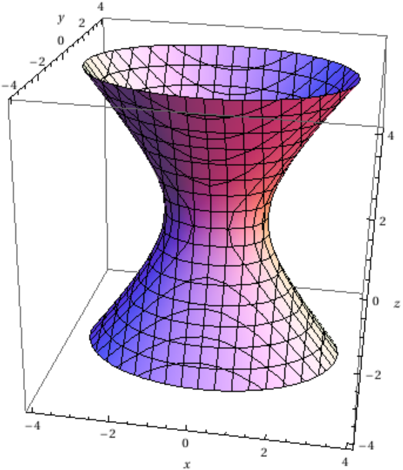
- In each plane $y = k$, the surface has a trace given by $\frac{x^2}{2} - \frac{z^2}{3} = 1 - k^2$: this is also the equation of a **hyperbola**

- In each plane $z = k$, the surface has a trace given by $\frac{x^2}{2} + y^2 = 1 + \frac{k^2}{3}$: this is the equation of an **ellipse**

The surface is a **hyperboloid**

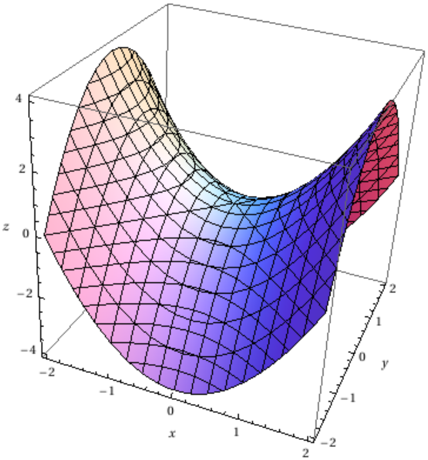
HYPERBOLOID

Surface given by the equation $\frac{x^2}{2} + y^2 - \frac{z^2}{3} = 1$: Graphical illustration



Hyperbolic paraboloid

Surface given by the equation $x^2 - y^2 = z$: Graphical illustration



Useful resource

Wolfram Alpha

www.wolframalpha.com