1 Conformality

1.1 Preservation of angle

The open mapping theorem tells us that an analytic function such that $f'(z_0) \neq 0$ maps a small neighborhood of z_0 onto a neighborhood of $f(z_0)$ in a one-to-one fashion. In particular, f maps continuously differentiable arcs through z_0 onto continuously differentiable arcs through $f(z_0)$. We now show that f preserves angles between two such arcs.

Suppose f is a complex function (not necessarily analytic) defined on a neighborhood of z_0 and such that $f(z) \neq f(z_0)$ for all $z \neq z_0$ in that neighborhood. If there exists $w = e^{i\varphi} \in \mathbb{C}$, with $\varphi \in \mathbb{R}$ such that for all $\theta \in \mathbb{R}$ and $r \in \mathbb{R}_+^*$,

$$\frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} \xrightarrow[r \to 0^+]{} e^{i\varphi} e^{i\theta}$$

then we say that f *preserves angles at* z_0. Indeed, f preserves the local angle between two arcs going through z_0, as shown in Figure 1.

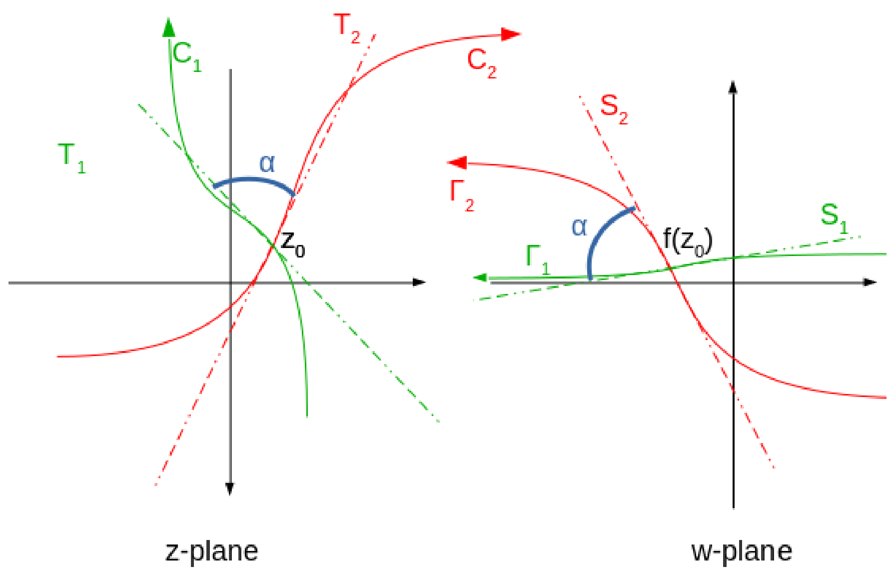

Figure 1: Two arcs C_1 and C_2 and there images Γ_1 and Γ_2 by the complex function f. The angle between the tangent lines T_1 and T_2 to C_1 and C_2 at z_0 is α. If f preserves angles at z_0, the angle between the tangent lines S_1 and S_2 to Γ_1 and Γ_2 at $f(z_0)$ is also α, in both magnitude *and* sign.

Theorem: Suppose f is analytic at z_0. Then f preserves angles at z_0 iff $f'(z_0) \neq 0$.

Proof: Let f be analytic in a neighborhood of z_0, and such that $f'(z_0) \neq 0$.

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \mathbb{R} \, , \lim_{r \to 0^+} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} = e^{i\theta} \lim_{r \to 0^+} \frac{\frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}}}{\frac{|f(z_0 + re^{i\theta}) - f(z_0)|}{r}} = e^{i\theta} \frac{f'(z_0)}{|f'(z_0)|}$$

So we see that if w exists, $w = f'(z_0)/|f'(z_0)|$.

Conversely, suppose that $f'(z_0) = 0$. If f is constant, f does not preserve angle. If f is not constant, there exists $N \in \mathbb{N}^*$ such that $f(z) = f(z_0) + (z - z_0)^N g(z)$, with g analytic at z_0, and $g(z_0) \neq 0$. In that case,

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \mathbb{R} \, , \, \frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} = e^{iN\theta} \frac{g(z_0 + re^{i\theta})}{|g(z_0 + re^{i\theta})|} \xrightarrow[r \to 0^+]{} e^{i\theta} e^{i(N-1)\theta} \frac{g(z_0)}{|g(z_0)|}$$

We see that f increases angle by a factor of N, so f does not preserve angles $\square$

Definition: A function which is analytic on an open connected set Ω and has a nonvanishing derivative is called a *conformal map*.

Examples: • exp maps any arbitrary vertical line $\{z : \Re(z) = x_0 \in \mathbb{R}\}$ onto the circle with center 0 and radius e^{x_0}. exp maps any horizontal line $\{z : \Im(s) = y_0 \in \mathbb{R}$ onto the open ray from 0 through e^{iy_0}. We see that exp preserves the orthogonality of these curves.

• The function $f(z) = z^2$ maps two curves crossing at 0 with angle α into two curves crossing at 0 with angle 2α. It is not conformal at $z = 0$.

Figure 2 gives an illustration of angle preservation from a global point of view. Although the blue and red squares are distorted by the map $f(z) = z^2 e^z$, the images of the blue and red squares by f still intersect with right angles.

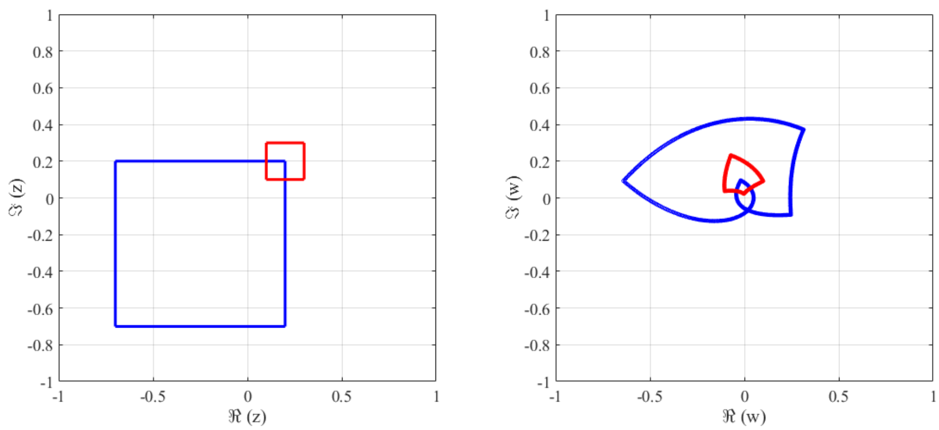

Figure 2: Two squares in the complex plane (left figure), and their image by the function $f(z) = z^2 e^z$ (right figure), which is conformal away from $z = -2$ and $z = 0$. The squares are fully distorted by the map, but the images still intersect at right angles.

Note: The definition of angle preservation contains the preservation of the magnitude of the angle *as well as* its orientation.
You can easily convince yourself that the mapping by the complex conjugate of an analytic function whose derivative does not vanish preserves the magnitude of the angle, but reverses the orientation. It is called an *indirectly conformal map*.

1.2 Length and area

• Consider an analytic function f such that $f'(z_0) \neq 0$.

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

Any small line segment with one end point at z_0 is expanded by an amount $|f'(z_0)|$. This expansion is independent of the direction of the line segment.

• Consider a set Ω in $\mathbb{R}^2$. Its area is given by

$$A(\Omega) = \iint_\Omega dxdy$$

The mapped set $\Omega' = f(\Omega)$, where $f = (u(x, y), v(x, y))$ is a bijective differentiable mapping, has an area given by

$$A(\Omega') = \iint_{\Omega'} dudv = \iint_\Omega |u_x v_y - u_y v_x| dxdy$$

Now, if $f(z) = u(x, y) + iv(x, y)$ is a conformal mapping on an open set containing Ω, f satisfies the Cauchy-Riemann relations, so that

$$|u_x v_y - u_y v_x| = |f'(z)|^2$$

and therefore

$$A(\Omega') = \iint_{\Omega} |f'(z)|^2 dxdy$$

Infinitesimal areas are expanded by the factor $|f'(z)|^2$.

We now turn to a fundamental class of conformal maps, known as linear fractional transformations, or Möbius transformations, which we have already introduced in Lecture 3.

2 Linear fractional transformations

2.1 Möbius transformation

As we have already seen in Lecture III, *Möbius transformations*, also called *linear fractional transformations*, are maps of the form

$$S(z) = \frac{az + b}{cz + d}$$

with $(a, b, c, d) \in \mathbb{C}^4$ such that $ad - bc \neq 0$ in order to avoid the situation in which S is a constant function. The domain of S is $\mathbb{C} \setminus \{-\frac{d}{c}\}$ if $c \neq 0$, and $\mathbb{C}$ if $c = 0$.

One often defines S on the extended complex plane $\hat{\mathbb{C}}$ by setting

$$S\left(-\frac{d}{c}\right) = \infty \qquad , \qquad S(\infty) = \frac{a}{c} \text{ if } c \neq 0 \quad , \quad S(\infty) = \infty \text{ if } c = 0$$

For all $z \in \mathbb{C} \setminus \{-\frac{d}{c}\}$,

$$S'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

so S is a conformal map on its domain.

Lastly, we have already seen that $\forall\ (a, b, c, d) \in \mathbb{C}^4$ such that $ad - bc \neq 0$, S has an inverse:

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}$$

2.2 The linear group

Theorem: The set $\mathcal{M}$ of Möbius transformations is a group under composition. Any Möbius transformation is a composition of the following maps:

(1) Translation: $z \mapsto z + a$, with $a \in \mathbb{C}$ constant

(2) Scaling and rotation: $z \mapsto kz$, $k \in \mathbb{C}^*$ constant

(3) Inversion: $z \mapsto \frac{1}{z}$

• Before we discuss the group structure, let us prove the last point of the theorem. If $c \neq 0$

$$\frac{az + b}{cz + d} = \frac{bc - ad}{c^2(z + \frac{d}{c})} + \frac{a}{c}$$

which is the composition of a translation, an inversion, a scaling and rotation, and another translation.
If $c = 0$, $S(z) = \frac{a}{d}z + \frac{b}{d}$ is the composition of a scaling and rotation and a translation.

• Regarding the group structure of $\mathcal{M}$, we have already done most of the work:

- $S(z) = z$ is the identity
- Any S has an inverse S^{-1}
- If $S_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \in \mathcal{M}$ and $S_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2} \in \mathcal{M}$, then

$$S_1(S_2(z)) = \frac{Az + B}{Cz + D}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

which demonstrates that the composition is also associative.

The decomposition of any linear fractional transformation into a translation, a rotation and scaling, and an inversion is very useful to visualize the effect of the transformation. This can be understood as follows.

In section 3, we will present a general method for visualizing maps. However, for the simple transformations mentioned here (translation, rotation and scaling, and inversion) there is an even simpler, more intuitive way, based on the Riemann sphere and the stereographic projection we saw in Lecture 1:

- Translations can be interpreted as translations of the Riemann sphere
- Rotations can be interpreted as rotations of the Riemann sphere about the vertical, i.e. North–South axis
- Scalings can be interpreted as the result of raising the Riemann sphere along that vertical axis
- Inversions can be interpreted as rotations of the Riemann sphere about a horizontal axis

These interpretations are illustrated in a beautiful way in the following video:
`https://www.youtube.com/watch?v=0z1fIsUNhO4`.
Using these elementary building blocks, interpreting the effect of a Möbius transformation is a much easier task.

2.3 Uniqueness of Möbius transformations

Let z_1, z_2, and z_3 be distinct points in $\hat{\mathbb{C}}$. To these points, we associate the Möbius transformation

$$S(z) = \begin{cases} \frac{\frac{z-z_2}{z-z_3}}{\frac{z_1-z_2}{z_1-z_3}} & \text{if } (z_1, z_2, z_3) \in \mathbb{C}^3 \\ \frac{z-z_2}{z-z_3} & \text{if } z_1 = \infty \\ \frac{z_1-z_3}{z-z_3} & \text{if } z_2 = \infty \\ \frac{z-z_2}{z_1-z_2} & \text{if } z_3 = \infty \end{cases} \tag{1}$$

Observe that $\forall (z_1, z_2, z_3) \in \hat{\mathbb{C}}^3$, $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$. We now show that S is the unique such linear fractional transformation, with the following general result.

For any two sets of distinct complex numbers $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ in the extended complex plane, there exists a unique Möbius transform taking z_n to w_n, for $n \in \{1, 2, 3\}$. To prove this, we use the lemma below.

Lemma: A Möbius transform can have at most two fixed points, unless S is the identity map.

Proof of the lemma: A fixed point z_0 of S is such that

$$\frac{az_0 + b}{cz_0 + d} = z_0 \quad \Leftrightarrow \quad cz_0^2 + (d - a)z_0 - b = 0$$

This polynomial equation has at most two solutions, which concludes our proof.

Proof of the existence and uniqueness of the Möbius transformation: Let us call $S_{z_1 z_2 z_3}$ the Möbius transform defined in (1). $S_{z_1 z_2 z_3}$ maps $\{z_1, z_2, z_3\}$ to $\{1, 0, \infty\}$. The inverse map of S_{w_1,w_2,w_3} takes $\{1, 0, \infty\}$ to $\{w_1, w_2, w_3\}$. Hence, $S^{-1}_{w_1,w_2,w_3} \circ S_{z_1 z_2 z_3}$ takes $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$.
Now, let us imagine that there are two linear fractional transformations S and T sending $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$. Then, $\forall n \in \{1, 2, 3\}$, $S \circ T^{-1}(w_n) = S(z_n) = w_n$. $S \circ T^{-1}$ has three fixed points, so $S = T$ □

Example: Find the Möbius transformation T that maps the points $\{-1, 0, 1\}$ to the points $\{-i, 1, i\}$.

From $T(0) = 1$, we find $d = b$. Imposing $T(-1) = -i$ and $T(1) = i$ then leads to a system of equations in which we find $a = ib$ and $c = -ib$, so that

$$T(z) = \frac{-z + i}{z + i}$$

2.4 Circlelines

Proposition: Let r and s be real numbers, and $k \in \mathbb{C}$. The equation $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$

- represents a line if $r = 0$ and $k \neq 0$
- represents a circle if $r \neq 0$ and $|k|^2 \geq rs$, with equation

$$\left|z + \frac{k}{r}\right| = \frac{1}{r}\sqrt{|k|^2 - rs}$$

This result can be easily proved by writing $z = x + iy$ and expanding in x and y.

Definition: The locus of the points of $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$, if non-empty, is called a *circleline*.

Note: The definition above may feel a bit odd at first, in the sense that it tries to combine under the same term two different objects: lines and cirles.
The reason why the definition makes sense in our context is that both circles and extended lines in the complex plane correspond to circles on the Riemann sphere, as discussed in Lecture I. Some authors, including Ahlfors, do not even use the term circleline, and call the locus of the points above a circle.

Lemma: Let $r \in \mathbb{R}$, and $(z_1, z_2) \in \mathbb{C}^2$, with $z_1 \neq z_2$. The locus of the equation $|z - z_1| = r|z - z_2|$ represents a circle if $r \neq 1$, and a line if $r = 1$, namely the line that is perpendicular to the line segment $[z_1, z_2]$ and passes through its midpoint.

Theorem: A Möbius transformation maps a circleline to a circleline.

Proof: Since any Möbius transformation is the composition of a translation, a scaling and rotation, and an inversion, we just have to show that the theorem holds independently for each of these transformations.

• The image of $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$ under the translation $z \mapsto w = z + a$ is

$$r|w - a|^2 + \overline{k}(w - a) + k\overline{w - a} + s = 0 \quad \Leftrightarrow \quad r|w|^2 + (\overline{k - a})w + (k - a)\overline{w} + r|a|^2 - (\overline{k}a + k\overline{a}) + s = 0$$

The last three terms are real numbers, so this is indeed the equation of a circleline.

• The result is immediate for a scaling and rotation, which corresponds to multiplication by a nonzero complex number

• For the inversion, we distinguish the case in which the circleline is a circle, and the case in which the circleline is a line

(i) Say the circleline is a circle, with equation $|z - a| = r$. If $a \neq 0$, the image of this circle under inversion is

$$\left|\frac{1}{w} - a\right| = r \quad \Leftrightarrow \quad \left|w - \frac{1}{a}\right| = \frac{r}{|a|}|w|$$

and from the previous lemma we know that the equation on the right-hand side is the equation of a circleline.

If $a = 0$, the image of $|z| = r$ under inversion is the circle $|w| = 1/r$.

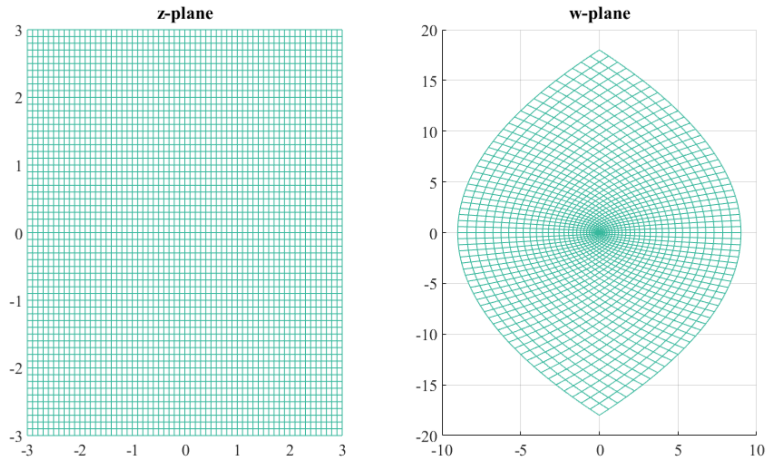

Figure 3: Mesh in the z-plane and its image by the map $z \mapsto z^2$

(ii) Let us now consider the line $\overline{k}z + k\overline{z} + s = 0$. Under inversion, this becomes

$$\frac{\overline{k}}{w} + \frac{k}{\overline{w}} + s = 0 \quad \Leftrightarrow \quad s|w|^2 + \overline{k}\overline{w} + kw = 0$$

which is a circleline $\square$

3 Visualizing maps

A streamlined way to visualize the transformation of a set by a map is to consider the effect of the map on a mesh of the set. This is what we do below.

• $f(z) = z^2$
We know that f is conformal on $\mathbb{C} \setminus \{0\}$. We consider the mesh

$$\begin{cases} z_{x_0} = x_0 + iy \\ z_{y_0} = x + iy_0 \end{cases}$$

for a countable set of x_0 and y_0. Any vertical line $z_{x_0} = x_0 + iy$ is mapped to

$$w = z_{x_0}^2 = u_{x_0} + iv_{x_0} = x_0^2 - y^2 + 2ix_0y$$

We see that the real and imaginary parts satisfy

$$v_{x_0}^2 = 4x_0^2y^2 = 4x_0^2(x_0^2 - u_{x_0})$$

This is the equation of a parabola in the w-plane, with focus $(0,0)$ and pointed in the negative direction.

Any horizontal line $z_{y_0} = x + iy_0$ is mapped to $w = x^2 - y_0^2 + 2ixy_0$. Hence

$$v_{y_0}^2 = 4y_0^2(u_{y_0} + y_0^2)$$

This is the equation of a parabola in the w-plane, with focus $(0,0)$ and pointed in the positive direction.
A mesh in the z-plane and its maps in the w-plane by the function $f(z) = z^2$ is shown in Figure 3.

• *The Cayley transform*
Consider the Möbius transformation

$$S(z) = \frac{z - i}{z + i}$$

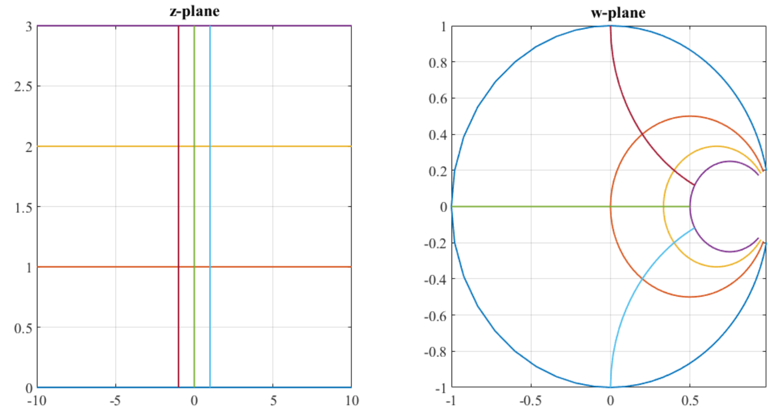

Figure 4: Mesh lines in the upper half of the z-plane and their images by the Cayley transform

sometimes called the Cayley transform. By direct computation, it is easy to see that S maps the real line to the unit circle. Another way to see this is that the points $\{\infty, 1, -1\}$ are mapped to $\{1, -i, i\}$. Since S maps circlelines to circlelines, we have the desired result.
Using the same theorem, it is easy to see that every horizontal line $z = x + iy_0$, with $y_0 > 0$ is mapped to a circle inside the unit circle. By direct computation, one finds that this is a circle of center $(y_0/(y_0+1), 0)$ and radius $1/(y_0+1)$.
Likewise, every vertical line $z = x_0 + iy$ is mapped to a circle of center $(1, -1/x_0)$ and radius $1/x_0$. The arcs of these circles corresponding to $y > 0$ are inside the unit disk.
The Cayley transform maps the upper half plane onto the unit disk.
A few mesh lines in the upper half of the z-plane and their maps in the w-plane by the Cayley transform are shown in Figure 4.

- *The Joukowski map*

A mapping which was historically important in fluid dynamics is the Joukowski map, defined by

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

This map is conformal everywhere on $\mathbb{C}$ except at $z = 0$ where is it not defined, and at $z = \pm 1$, where $dw/dz = 0$.

Observe that since $f(z) = f(1/z)$, two points $(z_1, z_2) \in \mathbb{C}^2$ such that $z_1 z_2 = 1$ are mapped onto the same image by f. f is two-to-one, and only one-to-one in a domain Ω if there are no two points z_1 and z_2 such that $(z_1, z_2) \in \Omega^2$ and $z_1 z_2 = 1$. f is for example one-to-one in $D_1(0)$, and in the exterior of $D_1(0)$.

To further visualize the map, let us consider families of circles $z = r_0 e^{i\theta}$, with $r_0 > 0$.

$$w = u + iv = \frac{1}{2}\left(r_0 e^{i\theta} + \frac{1}{r_0} e^{-i\theta}\right) \qquad \Rightarrow \qquad u = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos\theta \qquad , \qquad v = \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin\theta$$

Hence, the circle $C_{r_0}(0)$, $r_0 < 1$ is mapped onto an ellipse, and this ellipse degenerates into the line segment $[-1, 1]$ as $r_0 \to 1$.
Observe that $D_1(0)$ is mapped to $\mathbb{C} \setminus \{[-1, 1]\}$, and since $f(z) = f(1/z)$, the exterior of $D_1(0)$ is also mapped to $\mathbb{C} \setminus \{[-1, 1]\}$. The images of circles with radius $r_0 \geq 1$ and the images of the rays $x = \pm y$ perpendicular to the circles under the Joukowski map are shown in Figure 5 to illustrate this.
This mapping has relevance in fluid dynamics because the image of certain circles not centered at the origin under the map resembles the cross section of an airplane wing. You can for instance see the image of the circle $z = 0.1 + 0.2i + 0.85e^{i\theta}$, $\theta \in [0, 2\pi]$ under the Joukowski map in Figure 6.
The idea, then, at a time when numerical solvers for fluid dynamics were not available or very slow, was to compute the tractable problem of fluid potential flow around a cylinder with circular cross section and apply

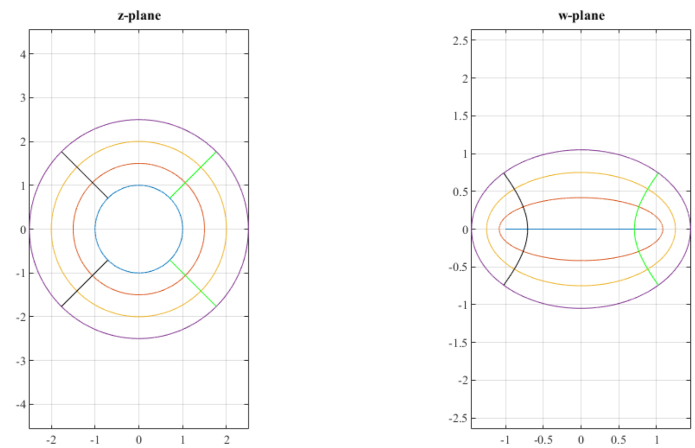

Figure 5: (Left) Circles $z = r_0 e^{i\theta}$, $\theta \in [0, 2\pi]$ in the z-plane, with $r_0 \in \{1, 1.5, 2, 2.5\}$ and rays $x = \pm y$, and (Right) their images in the w-plane under the Joukowski map $J(z) = 1/2(z + 1/z)$

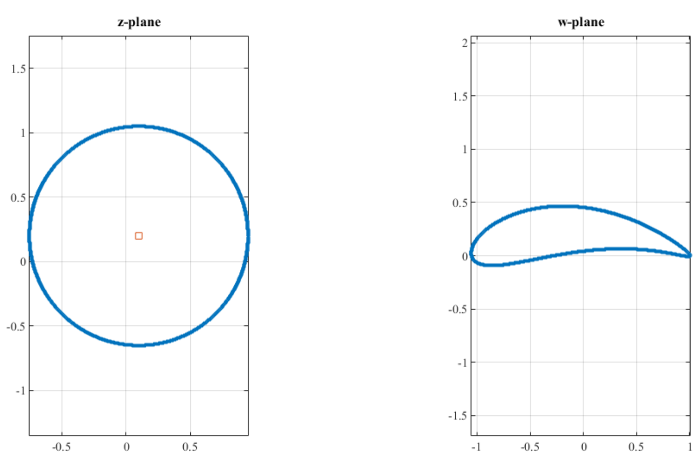

Figure 6: (Left) Circle $z = 0.1 + 0.2i + 0.85e^{i\theta}$, $\theta \in [0, 2\pi]$ in the z-plane (its center is shown with the red square marker), and (Right) its image in the w-plane under the Joukowski map $J(z) = 1/2(z + 1/z)$

the Joukowski mapping to the solution to obtain the fluid flow around the airplane wing. We will look at this in the next lecture.

4 The Riemann mapping theorem

Theorem (Riemann mapping theorem): Given any simply connected open set Ω which is not the whole plane, and a point z_0 in Ω, there exists a unique analytic function f in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that f defines a one-to-one mapping of Ω onto the disk $|w| < 1$.

The proof of this theorem is beyond the scope of this course. The detailed treatment of the theorem and of its proof is an important part of the syllabus of Complex Variables II, taught in the spring. The theorem and its proof are also covered in the one-term Complex Variables course.

Nevertheless, it is important for you at this stage to be aware of the existence of a one-to-one mapping onto the unit disk for *any* simply connected open set of $\mathbb{C}$ except for $\mathbb{C}$ itself. This result motivates the applications of conformal mapping we will present in the next lecture.

5 Conformal mapping of an annulus

The Riemann mapping theorem applies to simply connected open sets. Topologically, it is intuitive that the theorem cannot apply to non-simply connected domains: a hole cannot be made to disappear under a one-to-one continuous mapping.
One may however ask if one can generalize the Riemann mapping theorem to non-simply connected domains by looking into mappings of non-simply connected open sets to the simplest non-simply connected set one can think of, i.e. the annulus. Even more simply, one can ask whether any two concentric annuli are conformally equivalent. The answer to this question is, perhaps surprisingly, negative.

Theorem: $A(r_1, R_1) := \{z : r_1 < |z| < R_1\}$ and $A(r_2, R_2) := \{z : r_2 < |z| < R_2\}$ are conformally equivalent iff $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.

Proof: Without loss of generality, let us rescale $A(r_1, R_1) = A_1$ and $A(r_2, R_2) = A_2$ such that $A_1 = A(1, R_1)$ and $A_2 = A(1, R_2)$. Let us assume there is an analytic conformal map f such that $f(A_1) = A_2$. Then, since f is a homeomorphism between A_1 and A_2, either $|f(z)| \to 1$ as $|z| \to 1$ or $|f(z)| \to R_2$ as $|z| \to 1$. If the second situation holds, we can always define $g := R_2/f$ which is such that $|g(z)| \to 1$ as $|z| \to 1$. Hence, without loss of generality, we assume that $\lim_{|z|\to 1} |f(z)| = 1$ and thus $\lim_{|z|\to R_1} |f(z)| = R_2$. Now, since $f(z) \neq 0$ $\forall z \in A_1$, $\ln|f|$ is harmonic in A_1. Let

$$u(z) := \ln|f(z)| - \alpha \ln|z| \ , \ \ \alpha = \frac{\ln R_2}{\ln R_1} \quad , \quad \forall z \in A_1$$

Observe that

$$\lim_{|z|\to 1} u(z) = 0 = \lim_{|z|\to R_1} u(z)$$

We can extend u to a continuous function on $\overline{A}_1$, with $u = 0$ on ∂A_1. Since u is harmonic in A_1, $u \equiv 0$ on A_1. Therefore,

$$\forall z \in A_1 \ , \ |f(z)| = |z|^{\alpha} \tag{2}$$

Now, take $z_0 \in A_1$, and $r > 0$ such that $D_r(z_0) \subset A_1$. By the Maximum Modulus Principle, Eq.(2) implies that

$$\forall z \in D_r(z_0) \ , \ f(z) = e^{i\theta_0} z^{\alpha} \tag{3}$$

for some $\theta_0 \in \mathbb{R}$.
Taking the logarithmic derivative of Eq.(3), we find

$$\forall z \in D_r(z_0) \ , \ \ \frac{f'(z)}{f(z)} = \frac{\alpha}{z}$$

And since this is true for any $z_0 \in A_1$ (provided r is chosen small enough that $D_r(z_0) \subset A_1$)

$$\forall z \in A_1 \ , \ \frac{1}{2\pi i}\frac{f'(z)}{f(z)} = \frac{1}{2\pi i}\frac{\alpha}{z}$$

We integrate this equality over the circle of center 0 and radius $R_1^{1/2}$. The right-hand side gives α. The left-hand side is

$$\frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w} = \pm 1$$

and so it must be that $\alpha = 1$, i.e. $R_1 = R_2$.

Conversely, consider $A_1(r_1, R_1)$ and $A_2(r_2, R_2)$ such that $\frac{R_1}{r_1} = \frac{R_2}{r_2} = \beta$. Let $\gamma = r_2/r_1$. The mapping $z \mapsto \gamma z$ transforms $A_1(r_1, R_1) = A_1(r_1, \beta r_1)$ into $A(\gamma r_1, \gamma\beta r_1) = A_2(r_2, R_2)$.
This concludes our proof.

Note finally that it can be shown that any doubly connected region of $\mathbb{C}$ can be conformally mapped to an annulus. The proof of this result is however beyond the scope of this class.
From what we have just seen, the annulus to which the doubly connected region is mapped has a ratio r/R which is uniquely specified.

6 Conformal mapping of the unit disk to polygons

6.1 Mapping the unit disk to a polygon

The Riemann mapping theorem tells us that one can map any polygon to the unit disk. What is interesting in that particular case, is that the inverse map, from the unit disk to the polygon, has an explicit formula, called the Schwarz-Christoffel formula, which we give below without proof.

Theorem (Schwarz-Christoffel Formula): The functions $z = F(w)$ which map $D_1(0)$ conformally onto polygons with angles $\pi\alpha_k$ with $k \in [\![1, n]\!]$ in counterclockwise order are of the form

$$F(w) = A\int_0^w \prod_{k=1}^{n} (\zeta - w_k)^{-\beta_k}\, d\zeta + B$$

where A and B are complex constants, $\beta_k = 1 - \alpha_k$ such that $\sum_{k=1}^n \beta_k = 2$, and $(w_k)_{k=1,\ldots,n}$ n points on the unit circle.

The $(w_k)_{k=1,\ldots,n}$ are called the prevertices: they are the points on the unit circle such that $F(w_k) = z_k$ are the vertices of the polygon.
Now, observe that given two triplets of points on the unit circle $\{w_1, w_2, w_3\}$ and $\{w_1', w_2', w_3'\}$, there exists a Möbius transformation which maps the unit disk to the unit disk, and $\{w_1, w_2, w_3\}$ to $\{w_1', w_2', w_3'\}$. That means that we are always free to choose three of the w_k as we like, provided we are consistent with the ordering of the points.
Hence, for $n \leq 3$, the Schwarz-Christoffel formula can be viewed as an explicit formula for the mapping. A is easily determined by scaling and rotating the polygon appropriately, and B by mapping the origin of the disk to the desired point.
For $n \geq 4$, different choices of prevertices lead to different polygons, all consistent with the angles α_k. This is shown in Figure 7. In general, computing the non-free prevertices for a desired polygon must be done numerically. You may want to have a look at the great book *Schwarz-Christoffel Mapping* by T. Driscoll and L.N. Trefethen, Cambridge University Press, for a description of numerical algorithms and nice examples.

6.2 Generalizing the result: mapping the upper half plane to a polygon

We have seen that the Cayley transform

$$S(z) = \frac{z - i}{z + i}$$

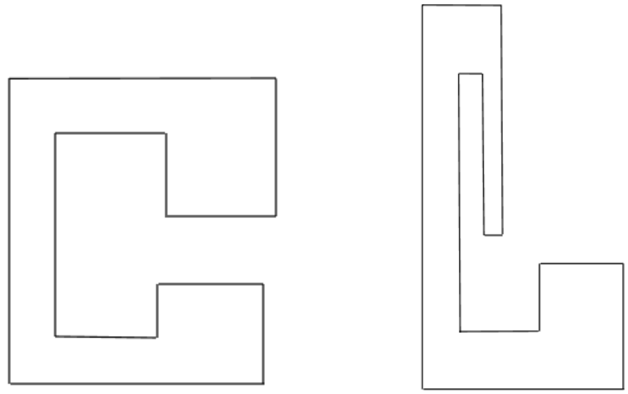

Figure 7: Two possible images of the unit disk for different choices of the prevertices, and the same choice of angles

maps the upper half plane to the unit disk. Its inverse is

$$z = S^{-1}(w) = i\frac{1+w}{1-w}$$

Composing F with S, we could have a mapping from the upper half plane to polygons. Defining

$$u = S^{-1}(\zeta) = i\frac{1+\zeta}{1-\zeta}$$

$$(\zeta - w_k)^{-\beta_k} = \left(\frac{u-i}{u+i} - w_k\right)^{-\beta_k} = \left(\frac{1-w_k}{u+i}\right)^{-\beta_k}\left[u - i\frac{1+w_k}{1-w_k}\right]^{-\beta_k}$$

We let $v_k := i(1+w_k)/(1-w_k)$, and observe that

$$d\zeta = \frac{2i}{(u+i)^2}du$$

to obtain the general form of the mapping from the upper half plane to polygons:

$$G(z) := F \circ S(z) = 2iA\int_0^z \prod_{k=1}^n \left(\frac{1-w_k}{u+i}\right)^{-\beta_k}(u-v_k)^{-\beta_k}\frac{du}{(u+i)^2} + B$$

Using the fact that $\sum_{k=1}^n \beta_k = 2$, this can be written in the more concise form

$$G(z) = \alpha\int_0^z \prod_{k=1}^n (u-v_k)^{-\beta_k}du + \beta$$

with α and β constants, and β_k as before.

6.3 Examples

•*Mapping the upper half plane to the semi-infinite strip* $-\pi/2 < \Re(w) < \pi/2$, $\Im(w) > 0$
We have $w_1 = -\pi/2$, $w_2 = \pi/2$. Let us choose $v_1 = -1$, $v_2 = 1$. The mapping is given by

$$G(z) = \alpha\int_0^z (u+1)^{-1/2}(u-1)^{-1/2}du + \beta = \alpha\int_0^z \frac{1}{\sqrt{u^2-1}}du + \beta = \alpha' \arcsin z + \beta$$

Setting $G(-1) = -\pi/2$ and $G(1) = \pi/2$ leads to the system

$$\begin{cases} -\frac{\pi}{2}\alpha' + \beta = -\frac{\pi}{2} \\ \frac{\pi}{2}\alpha' + \beta = \frac{\pi}{2} \end{cases}$$

from which we conclude that $\beta = 0$ and $\alpha' = 1$: $f(z) = \arcsin z$ is a map from the upper half plane to the desired semi-infinite strip.

• *Mapping the upper half plane to a rectangle*

Let us assume the rectangle is rotated and translated so that its vertices are $w_1 = -K_1 + iK_2$, $w_2 = -K_1$, $w_3 = K_1$, and $w_4 = K_1 + iK_2$. By symmetry, we choose the prevertices such that $v_1 = -k^{-1/2}$, $v_2 = -1$, $v_3 = 1$, and $v_4 = k^{-1/2}$, where k represents the fact that only 3 points can be freely specified.

We thus have

$$w = G(z) = \beta + \alpha \int_0^z \prod_{k=1}^{4} (u - v_k)^{-1/2} = \alpha \int_0^z \frac{du}{\sqrt{u^2 - \frac{1}{k}}\sqrt{u^2 - 1}}$$

where we have set $\beta = 0$ so that $F(0) = 0$. This is an elliptic integral of the first kind, which can be written in the more concise form

$$w = G(z) = \alpha \int_0^{\arcsin z} \frac{d\theta}{\sqrt{1 - k \sin^2 \theta}}$$

The constant α represents the freedom in rotating and scaling the rectangle, and the constant k must be varied to obtain the desired aspect ratio K_2/K_1 for the rectangle.