

1 The Euler gamma function

The Euler gamma function is often just called the gamma function. It is one of the most important and ubiquitous special functions in mathematics, with applications in combinatorics, probability, number theory, differential equations, etc. Below, we will present all the fundamental properties of this function, and prove that they all naturally follow from its integral representation.

1.1 Definition

Theorem (the Euler gamma function): There exists a unique function Γ on \mathbb{C} such that:

- (a) Γ is meromorphic on \mathbb{C}
- (b) $\forall n \in \mathbb{N}, \Gamma(n + 1) = n!$
- (c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- (d) $\forall s \in \mathbb{C}$ such that $\Re(s) > 0$

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx$$

- (e) $\forall s \in \mathbb{C}$ except for poles

$$\Gamma(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^{\infty} e^{-x} x^{s-1} dx$$

- (f) $\forall s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

where

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n\right)$$

is called the *Euler constant*

- (g) $\forall s \in \mathbb{C}$ except for poles

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n!n^s}{s(s+1)\dots(s+n)}$$

- (h) Γ has no zeros; in other words, $1/\Gamma$ is an entire function
- (i) The poles of Γ are the nonpositive integers $s = 0, -1, -2, \dots$. The pole of Γ at $s = -n$, with $n \in \mathbb{N}$ is a simple pole, with residue

$$\text{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$$

- (j) $\forall s \in \mathbb{C}$ except for poles, $\Gamma(s + 1) = s\Gamma(s)$
- (k) $\forall s \in \mathbb{C}$ except for poles, $\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)}$

1.2 Proving the properties

Let us start with the integral representation, which may be viewed as a definition for the function:

$$\forall s \in \mathbb{C} : \Re(s) > 0, \quad \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx$$

For $n \in \mathbb{N}^*$, let

$$f_n(s) := \int_0^n e^{-x} x^{s-1} dx$$

f_n is analytic, and

$$|\Gamma(s) - f_n(s)| \leq \int_n^{+\infty} x^{\Re(s)-1} e^{-x} dx$$

The right hand side converges uniformly in every half-plane $\Re(s) \geq \delta$ with $\delta > 0$, so Γ is analytic on $\Re(s) > 0$.

• For $\Re(s) > 0$, integration by parts immediately yields

$$\Gamma(s+1) = s\Gamma(s)$$

Furthermore, $\Gamma(1) = 1$, so by induction one readily finds

$$\Gamma(n+1) = n!$$

which proves (b).

• $\Gamma(1/2) = \int_0^{+\infty} e^{-x} x^{-1/2} dx = 2 \int_0^{+\infty} e^{-t^2} dt = \sqrt{\pi}$, which proves (c).

We can use the functional equation $\Gamma(s+1) = s\Gamma(s)$ to analytically continue Γ to a meromorphic function on \mathbb{C} . Indeed

$$\Gamma_1(s) := \frac{\Gamma(s+1)}{s}$$

is an analytic function on $\{s \in \mathbb{C} : \Re(s) > -1\} \setminus \{0\}$ such that $\Gamma_1(s) = \Gamma(s)$ for $\Re(s) > 0$. Furthermore, $s = 0$ is a simple pole of Γ_1 , with

$$\text{Res}_{s=0} \Gamma_1 = \Gamma(1) = 1$$

Likewise, for s such that $\Re(s) > -2$ and $s \neq 0, s \neq -1$, define

$$\Gamma_2(s) = \frac{\Gamma_1(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)}$$

which is analytic on $\{s \in \mathbb{C} : \Re(s) > -2\} \setminus \{-1, 0\}$ and coincides with Γ_1 for $\Re(s) > -1$. $s = -1$ is a simple pole of Γ_2 , with residue -1.

By induction, if we have Γ_{n-1} as the analytic continuation of Γ to $\Re(s) > 1-n$, $s \notin \{-n+2, -n+3, \dots, -2, -1, 0\}$, then we define

$$\Gamma_n(s) := \frac{\Gamma_{n-1}(s+1)}{s} = \frac{\Gamma(s+n)}{s(s+1)\dots(s+n-1)}$$

which is a meromorphic function for $\Re(s) > -n$, with poles $s = -n+1, -n+2, \dots, -1, 0$ and residue $\frac{(-1)^n}{n!}$ at $s = -n$.

• Let us now write

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1} dx + \int_1^{+\infty} e^{-x} x^{s-1} dx$$

The second term on the right-hand side is analytic for all $s \in \mathbb{C}$. We call the first term F . We have

$$F(s) := \int_0^1 e^{-x} x^{s-1} dx = \int_0^1 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^{n+s-1} dx = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \int_0^1 x^{n+s-1} dx$$

where we have interchanged the order of summation and integration using absolute convergence. We thus get the final expression:

$$\Gamma(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^{+\infty} e^{-x} x^{s-1} dx$$

as given in item (e), which has the desirable property of highlighting the poles of Γ .

• Consider $\lim_{n \rightarrow \infty} \int_0^n (1-x/n)^n x^{s-1} dx$ for $\Re(s) > 0$.

As $n \rightarrow \infty$,

$$\left(1 - \frac{x}{n}\right)^n x^{s-1} \rightarrow e^{-x} x^{s-1}$$

pointwise. Furthermore, $\forall n \in \mathbb{N}, \forall x \in [0, n]$,

$$\left(1 - \frac{x}{n}\right)^n \leq e^{-x}$$

Hence, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^{+\infty} \left(1 - \frac{x}{n}\right)^n x^{s-1} dx = \int_0^{+\infty} e^{-x} x^{s-1} dx = \Gamma(s)$$

Now, for $\Re(s) > 0$, one can show by induction that

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx = \frac{n! n^s}{s(s+1) \dots (s+n)}$$

This is how it goes. The property holds for $n = 1$:

$$\int_0^1 (1-x)x^{s-1} dx = \frac{1}{s(s+1)}$$

Let us assume it holds for $n - 1$. Then

$$\begin{aligned} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx &= n^s \int_0^1 (1-t)^n t^{s-1} dt = \frac{n^s}{s} \left\{ [(1-t)^n t^s]_0^1 + n \int_0^1 (1-t)^{n-1} t^s dt \right\} \\ &= \frac{n^{s+1}}{s} \int_0^1 (1-t)^{n-1} t^s dt = n^s \frac{n}{s} \int_0^1 (1-t)^{n-1} t^{(s+1)-1} dt \\ &= \frac{n! n^s}{s(s+1) \dots (s+n)} \end{aligned}$$

where we have used the induction hypothesis for the last step. Hence, for $\Re(s) > 0$,

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n! n^s}{s(s+1) \dots (s+n)}$$

We want to extend this result to \mathbb{C} , excluding the poles of Γ . Let us consider the function $1/\Gamma$. For $\Re(s) > 0$,

$$\begin{aligned} \frac{1}{\Gamma(s)} &= \lim_{n \rightarrow \infty} \frac{s(s+1) \dots (s+n)}{n! n^s} = s \lim_{n \rightarrow \infty} e^{-s \ln n} (1+s)(1+\frac{s}{2}) \dots (1+\frac{s}{n}) \\ &= s \lim_{n \rightarrow \infty} e^{s(\sum_{k=1}^n \frac{1}{k} - \ln n)} \prod_{k=1}^n \left(1 + \frac{s}{k}\right) e^{-s/k} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \end{aligned}$$

From the Weierstrass factorization theorem, we know that this represents an entire function with zeros at the nonpositive integers, which proves (f) and (h).

• It remains to prove (k). Away from the poles of Γ , one can write

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = -\frac{1}{s\Gamma(s)\Gamma(-s)} = -\frac{1}{s} s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} (-s) e^{-\gamma s} \prod_{n=1}^{\infty} \left(1 - \frac{s}{n}\right) e^{s/n} = s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right) = \frac{\sin(\pi s)}{\pi}$$

where the last equality follows from the example we treated in class in the last lecture.

1.3 Volume of an n-dimensional ball

Consider the function of n real variables

$$f(x_1, x_2, \dots, x_n) = \exp\left(-\frac{1}{2} \sum_{k=1}^n x_k^2\right)$$

We can evaluate

$$\int_{\mathbb{R}^n} f d\mathbf{x} = \prod_{k=1}^n \left(\int_{-\infty}^{+\infty} e^{-\frac{x_k^2}{2}} dx_k \right) = (\sqrt{2\pi})^n \quad (1)$$

Now, since f is rotationally symmetric, one can use generalized spherical coordinates to rewrite the integral as follows:

$$\int_{\mathbb{R}^n} f d\mathbf{x} = \int_0^{+\infty} e^{-\frac{r^2}{2}} \int_{S^{n-1}(r)} dA dr = \int_0^{+\infty} e^{-\frac{r^2}{2}} A_{n-1}(r) dr$$

where $S^{n-1}(r)$ is the $(n-1)$ -sphere of radius r , dA is the area element, and $A_{n-1}(r)$ is the surface area of the sphere $S^{n-1}(r)$.

Now, $A_{n-1}(r) = r^{n-1} A_{n-1}(1)$, so

$$\int_{\mathbb{R}^n} f d\mathbf{x} = A_{n-1}(1) \int_0^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr = 2^{\frac{n-2}{2}} A_{n-1}(1) \int_0^{+\infty} t^{\frac{n}{2}-1} e^{-t} dt = 2^{\frac{n-2}{2}} A_{n-1}(1) \Gamma\left(\frac{n}{2}\right) \quad (2)$$

Comparing Eq.(1) and (2), we obtain the equality:

$$A_{n-1}(r) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$$

Hence, $V_n(r)$, the volume of the n -ball of radius r is given by

$$V_n(r) = \int_0^r A_{n-1}(t) dt = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r t^{n-1} dt = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} r^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} r^n \quad (3)$$

We observe something remarkable: the volume of the unit ball increases for $n \leq 5$, but then decreases to 0 as $n \rightarrow \infty$. If one does not restrict n to be an integer, one can compute the max of V_n by setting $dV_n/dn = 0$, and obtain $n_{max} \approx 5.25694$.

2 The Riemann zeta function

Just like the gamma function, the Riemann zeta function plays a key role in many fields of mathematics. It is however much less well understood and characterized than the zeta function. There remains several open problems associated with it, including THE open problem of mathematics: the Riemann hypothesis.

2.1 Definition

Theorem (the Riemann zeta function): There exists a unique function ζ on \mathbb{C} such that:

- (a) ζ is meromorphic on \mathbb{C}
- (b) For $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- (c) For $\Re(s) > 1$, ζ also has the infinite product representation

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

where, as indicated, the product ranges over the prime numbers.

- (d) ζ has no zeros in the region $\Re(s) > 1$
- (e) ζ has no zeros on the line $\Re(s) = 1$
- (f) The zeros of ζ in the region $\Re(s) \leq 0$ are at $s = -2k$, $k \in \mathbb{N}^*$
- (g) ζ has a unique pole, at $s = 1$, with residue 1.
- (h) The values of ζ at even positive integers are given by Euler's formula:

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad n \in \mathbb{N}^*$$

where the B_k are the Bernoulli numbers, defined by the following Taylor expansion:

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m$$

- (i) ζ takes the following values for negative integers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad n \in \mathbb{N}^*$$

- (j) ζ satisfies the functional equation

$$\zeta^*(1-s) = \zeta^*(s)$$

where ζ^* is the *symmetrized zeta function* defined by

$$\zeta^*(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

- (k) $\forall s \in \mathbb{C} \setminus \{0, 1\}$

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = -\frac{1}{1-s} - \frac{1}{s} + \frac{1}{2} \int_1^{\infty} (t^{-\frac{s+1}{2}} + t^{\frac{s-2}{2}})(\theta(t) - 1) dt$$

where the function θ is one of the Jacobi theta series, defined as

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

- (l) $\forall s \in \mathbb{C} \setminus \{1\}$,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s dz}{e^z - 1} z$$

where C is the keyhole contour shown in Figure 1, with ϵ arbitrary as long as the circle does not enclose an integer multiple of $2\pi i$. The branch of the logarithm in the integrand is to be chosen such that $-\pi < \text{Arg}(-z) < \pi$.

- (m) *Connection to prime number enumeration*

Define $\psi(x) = \sum_{p^k \leq x} \ln p$, with p prime numbers.

ψ is called the *Von Mangoldt* weighted prime counting function. Then, for any noninteger $x > 1$,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln 2\pi$$

where the sum is over the zeros ρ of the Riemann zeta function.

The formula above has important consequences for prime number enumeration, provided one can locate the zeros ρ of ζ in the complex plane. For example, the fact that ζ has no zeros such that $\Re(s) \geq 1$ leads, after some work, to the prime number theorem given below.

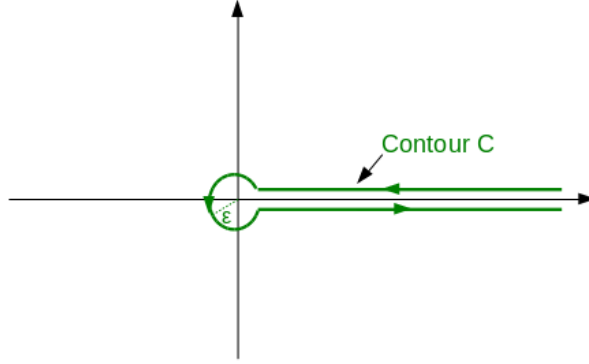


Figure 1: Contour C used for the integral representation of ζ in property (1).

Theorem (Prime number theorem): Let $\pi(x)$ denote the number of prime numbers less than or equal to x . We have

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{\left(\frac{x}{\ln x}\right)} = 1$$

If there is enough time at the end of the course, we will work out the details of the proof of this theorem based on the properties of the Riemann zeta function.

Of course, the exact location of the nontrivial zeros of the Riemann zeta function remains a key open problem. It is usually described as the *Riemann hypothesis*, which conjectures that all the nontrivial zeros of ζ are on the line $\Re(s) = \frac{1}{2}$, called the *critical line*.

2.2 Proving the properties

- We start with the standard definition of ζ :

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It is clear that this series converges absolutely for $\Re(s) > 1$, and the convergence is uniform on any half-plane $\Re(s) > \delta$ with $\delta > 1$. Hence ζ is analytic on $\Re(s) > 1$.

- Likewise

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

converges absolutely iff $\sum_{p \text{ prime}} |p^{-s}| = \sum_{p \text{ prime}} p^{-\Re(s)}$ converges, which happens for $\Re(s) > 1$. Hence

$$F(s) := \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

is analytic and nonzero in $\Re(s) > 1$. It remains to show that $\zeta(s) = F(s)$ on this set.

For $\Re(s) > 1$, let

$$\zeta_N(s) := \prod_{p \leq N, \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \leq N} \sum_{k=0}^{\infty} p^{-ks} = \sum_{\substack{n=p_1^{c_1} p_2^{c_2} \dots p_m^{c_m} \\ p_1, p_2, \dots, p_m \leq N \\ p_i \text{ prime}}} \frac{1}{n^s}$$

where the last equality was obtained by reorganizing terms in the absolutely convergent series. Hence, by the fundamental theorem of arithmetic,

$$|\zeta(s) - \zeta_N(s)| \leq \sum_{n > N} \frac{1}{n^s} \xrightarrow{N \rightarrow \infty} 0$$

which proves that for $\Re(s) > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

which is called the Euler product formula, and proves point (c).

- Property (d) follows immediately from the Euler product formula
- We now turn to the Mellin transform representation. We start by showing that θ satisfies

$$\forall t > 0, \theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

We start with $f(x) = \exp(-\pi t x^2)$, whose Fourier transform is

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx = \frac{1}{\sqrt{t}} \exp\left(-\frac{\pi k^2}{t}\right)$$

The Poisson summation formula then tells us

$$\theta(t) = \sum_{n=-\infty}^{n=+\infty} e^{-\pi n^2 t} = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{\pi k^2}{t}} = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

We observe that for $t > 0$,

$$\theta(t) - 1 = 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t} \leq 2 \sum_{n=1}^{\infty} e^{-\pi n t} = 2 \frac{e^{-\pi t}}{1 - e^{-\pi t}}$$

Thus, $\theta(t) = 1 + \mathcal{O}(e^{-\pi t})$ for $t \rightarrow \infty$. So using the equality $\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$, we conclude that

$$\theta(t) = \frac{1}{\sqrt{t}} \left(1 + \mathcal{O}(e^{-\pi/t})\right), \quad t \rightarrow 0^+ \quad \Rightarrow \quad \theta(t) = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad t \rightarrow 0^+$$

We are now ready to turn to a representation for ζ . For $\Re(s) > 1$, we write

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{+\infty} e^{-x} x^{s/2-1} dx \quad \Leftrightarrow \quad \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{+\infty} e^{-\pi n^2 t} t^{s/2-1} dt$$

Summing over n on both sides of the equality, we obtain

$$\zeta^*(s) = \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-\pi n^2 t} t^{s/2-1} dt = \int_0^{\infty} \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t} \right) t^{s/2-1} dt = \int_0^{\infty} \frac{\theta(t) - 1}{2} t^{s/2-1} dt$$

where we have used the estimates for θ to exchange the sum and integral signs. Now, let

$$g(t) := \frac{\theta(t) - 1}{2}$$

g satisfies the equality

$$g(t) = \frac{1}{\sqrt{t}} g\left(\frac{1}{t}\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2}$$

Thus,

$$\begin{aligned} \zeta^*(s) &= \int_0^1 g(t) t^{s/2-1} dt + \int_1^{\infty} g(t) t^{s/2-1} dt = \int_0^1 \left(\frac{1}{\sqrt{t}} g\left(\frac{1}{t}\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2} \right) + \int_1^{\infty} g(t) t^{s/2-1} dt \\ &= -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2} \int_1^{\infty} (\theta(t) - 1) (t^{-s/2-1/2} + t^{s/2-1}) dt \end{aligned}$$

which is the desired Mellin transform representation given in (k)

- The integral defines an entire function on \mathbb{C} , showing that ζ can indeed be continued to a meromorphic function on \mathbb{C} .

- The Mellin transform representation immediately yields $\zeta^*(1-s) = \zeta^*(s)$, which is property (j), and which can be rewritten as

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

- ζ^* has simple poles at $s = 0$ and at $s = 1$, with residues -1 and 1 . Therefore, $\zeta(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2})} \zeta^*(s)$ has a pole at $s = 1$ with residue $\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})} = 1$ and a pole at $s = 0$ with residue $1/\Gamma(0) = 0$. We see that the singularity at 0 is in fact a removable singularity, which proves (g).

- (4) shows that the zeros of ζ for $\Re(s) < 0$ are precisely $s = -2k$, $k \in \mathbb{N}^*$, which is property (f).

- Let us now prove (e): ζ has no zeros on the line $\Re(s) = 1$. Let $\sigma > 1$ and $t \in \mathbb{R}^*$ and consider the quantity

$$\begin{aligned} \mu &= \ln |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| = 3 \ln |\zeta(\sigma)| + 4 \ln |\zeta(\sigma + it)| + \ln |\zeta(\sigma + 2it)| \\ &= 3 \ln \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-\sigma}} \right) + 4 \ln \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-\sigma - it}} \right) + \ln \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-\sigma - 2it}} \right) \\ &= \sum_{p \text{ prime}} (-3 \ln |1 - p^{-\sigma}| - 4 \ln |1 - p^{-\sigma - it}| - \ln |1 - p^{-\sigma - 2it}|) \\ &= \sum_{p \text{ prime}} [-3 \Re(\text{Ln}(1 - p^{-\sigma})) - 4 \Re(\text{Ln}(1 - p^{-\sigma - it})) - \Re(\text{Ln}(1 - p^{-\sigma - 2it}))] \end{aligned}$$

where, as always, Ln is the principal branch of the logarithm. Our next step will be to use power series for Ln , which we can since

$$|p^{-\sigma}| < 1, \quad |p^{-\sigma - it}| < 1, \quad |p^{-\sigma - 2it}| < 1$$

For $s = a + ib$ such that $\Re(s) > 1$,

$$-\text{Ln}(1 - p^{-s}) = \sum_{k=1}^{+\infty} \frac{p^{-ks}}{k}$$

so that

$$-\Re(\text{Ln}(1 - p^{-s})) = \sum_{k=1}^{+\infty} \frac{p^{-ka}}{k} \cos(kb \ln p)$$

Therefore,

$$\mu = \sum_{p \text{ prime}} \sum_{k=1}^{+\infty} \frac{p^{-k\sigma}}{k} [3 + 4 \cos(kt \ln p) + \cos(2kt \ln p)] = 2 \sum_{p \text{ prime}} \sum_{k=1}^{+\infty} \frac{p^{-k\sigma}}{k} [1 + \cos(kt \ln p)]^2 \geq 0$$

We can therefore say that $e^\mu \geq 1$, which means that

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1 \quad (5)$$

All we now have to show is that this inequality prevents ζ from having a zero on the line $\Re(s) = 1$. Let us assume the contrary: $\exists t \in \mathbb{R}^*$ such that $\zeta(1 + it) = 0$. We then look at the asymptotic behavior of each term in (5) as $\sigma \rightarrow 1^+$:

$$\zeta(\sigma) \sim \frac{1}{s-1} \quad \zeta(\sigma + it) \sim K_1(\sigma - 1) \quad \zeta(\sigma + 2it) \sim K_2 \quad \text{as } \sigma \rightarrow 1^+, \quad (K_1, K_2) \in \mathbb{C}^2$$

where the first asymptotic estimate is tight, and the other two are conservative, in the sense that $\zeta(\sigma + it)$ could go to zero faster, and $\zeta(\sigma + 2it)$ could also go to zero as $\sigma \rightarrow 1^+$. We then obtain the following conservative asymptotic estimate as $\sigma \rightarrow 1^+$:

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \sim K_3(\sigma - 1) \quad \text{as } \sigma \rightarrow 1^+ \quad K_3 \in \mathbb{C}$$

This contradicts the result $|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \forall \sigma > 1$ and $t \in \mathbb{R}^*$. ζ does not have any zero with real part equal to 1.

Note that the miraculously simple trick used at the heart of our proof is often attributed to the Polish mathematician Franz Mertens.

- We conclude this lecture with a derivation of property (l).

The derivation starts with another useful identity. Let $s \in \mathbb{C}$ such that $\Re(s) > 1$.

$$\int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} dt = \int_0^{+\infty} t^{s-1} \sum_{n=1}^{+\infty} e^{-nt} dt = \sum_{n=1}^{+\infty} \int_0^{+\infty} t^{s-1} e^{-nt} dt = \sum_{n=1}^{+\infty} \frac{1}{n^s} \int_0^{+\infty} u^{s-1} e^{-u} du = \zeta(s)\Gamma(s)$$

where we have used absolute convergence to interchange the order of integration and summation.

Now, by Cauchy's theorem, the value of the integral

$$\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}$$

does not depend on the shape of the curve C , provided C does not enclose a pole of the integrand, i.e. a multiple of $2\pi i$. We are therefore free to choose the key hole contour shown in Figure 1, and to take the limit $\epsilon \rightarrow 0$ for that contour. It is then straightforward to verify that the contribution from the circle of radius ϵ tends to zero. When $\epsilon \rightarrow 0$, the only contributions to the integral thus come from the two extended branches of the contour C , and we have

$$\int_{+\infty}^0 \frac{\rho^{s-1} e^{-is\pi}}{e^\rho - 1} d\rho + \int_0^{+\infty} \frac{\rho^{s-1} e^{is\pi}}{e^\rho - 1} d\rho = (e^{is\pi} - e^{-is\pi}) \int_0^{+\infty} \frac{\rho^{s-1}}{e^\rho - 1} d\rho = 2i \sin(s\pi) \zeta(s) \Gamma(s)$$

It is a simple exercise to verify that the integral over the small circle tends to zero as ϵ tends to 0 when $\Re(s) > 1$. Hence, for $\Re(s) > 1$,

$$\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = 2i \sin(s\pi) \zeta(s) \Gamma(s) \Leftrightarrow \zeta(s) = \frac{1}{2i \sin(s\pi) \Gamma(s)} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} \quad (6)$$

We observe that the integral in (6) is an entire function of s , so (6) can be viewed as a way to analytically extend ζ to a meromorphic function in \mathbb{C} which is equivalent to the Mellin transform representation. Note that when $\Re(s) \leq 1$, it is not true anymore that the contribution from the circle of radius ϵ tends to 0 as $\epsilon \rightarrow 0$.

- Property (i) follows from (6). This is left as a straightforward exercise, as well as property (h) which follows from property (i) and the functional equation.