1 Brief review of limits, continuity, and differentiation

With the stereographic projection, we have just seen, without clearly stating it, our first example of a function which takes real numbers as inputs, and outputs a complex number.

As we move on to studying functions and their properties, 4 cases may in principle be considered: real functions of real variables, real functions of complex variables, complex functions of real variables, and complex functions of complex variables. Fortunately, the vast majority of concepts we will apply to functions can be defined in the same way in the 4 cases. This is because the concept of limit can be defined in the same way in all 4 cases.

1.1 Limits

Definition: A function f has the limit L (L finite) as z tends to z_0, written $\lim_{z\to z_0} f(z) = L$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ for all z such that $0 < |z - z_0| < \delta$.

You can observe that this indeed agrees with the definition you have all seen for real functions of real variables. If the input z is complex, or the output $f(z)$ is complex, what used to be the absolute value should now be understood as the modulus.

The situation in which we are interested in the limit $z \to \infty$ is as follows. We call an *ϵ-neighborhood of ∞* the set

$$\{z \in \mathbb{C} : \frac{1}{|z|} < \epsilon\}$$

Definition: If there is $L \in \mathbb{C}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{|z|} < \delta \quad \Rightarrow \quad |f(z) - L| < \epsilon$$

then the limit as z approaches ∞ of f is L, and we write $\lim_{z\to\infty} f(z) = L$

Similar definitions are easily constructed for the cases in which L is infinite, as you have done for real variables.

• As we have seen in the previous lecture, for two complex numbers z_1 and z_2, $|z_1 z_2| = |z_1||z_2|$ and $|z_1 + z_2| \leq |z_1| + |z_2|$ so recalling the proofs in the real variables case, we easily see that the limit laws (sum law, product law) also hold in the complex case.

• Since $|z| = |\overline{z}|$ for any $z \in \mathbb{C}$, if $\lim_{z\to z_0} f(z) = L$, then $\lim_{z\to z_0} \overline{f(z)} = \overline{L}$

• Combining the previous two results,

$$\begin{cases} \lim_{z\to z_0} \Re(f(z)) = \Re(L) \\ \lim_{z\to z_0} \Im(f(z)) = \Im(L) \end{cases} \tag{1}$$

which can be seen as an alternative way of defining the limit of $f(z)$.

Examples: $\lim_{z\to 3} z^3 + 1 = 28$

$\lim_{z\to\infty} \frac{3z^3+2z^2+1}{z^3-5} = 3$

$\lim_{z\to 1-2i} \Im(z) = -2$

The function $f(z) = \frac{\Re(z)\Im(z)}{z^2}$ does not have a limit as $z \to 0$

1.2 Continuity

Definition: A function f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

This is again the same definition as in the case of real variables so we know that we could easily prove that the sum and product of continuous functions are continuous functions.

From the definition of the limit, we can also conclude that if f is continuous at z_0, then so is $\overline{f}$, $\Re(f)$ and $\Im(f)$.

Examples: $f(z) = z$ is continuous on $\mathbb{C}$.
It is therefore clear that any polynomial $P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$ is continuous on $\mathbb{C}$.
Moreover, any rational function $R(z) = P(z)/Q(z)$ with $P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$ and $Q(z) = d_m z^m + d_{m-1} z^{m-1} + \ldots + d_1 z + d_0$ polynomials, is continuous wherever it is defined, i.e. on all of $\mathbb{C}$ except for the points for which $Q(z) = 0$.

1.3 Derivatives

Definition: A function f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This number is called the *derivative of* f *at* z_0, written

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Once more, the definition is standard. However, depending on the case considered – real function or complex function, real variable or complex variable – the existence of a derivative can have far-reaching consequences regarding the properties of the function.

• Let us start with the simplest case: a complex function f of a real variable x. One may write

$$f(x) = u(x) + iv(x)$$

f has a derivative $f'(x_0)$ at x_0 if and only if u and v are differentiable at x_0, and

$$f'(x_0) = u'(x_0) + iv'(x_0)$$

• Consider now a *real* function f of a *complex* variable z. If f is differentiable at z_0, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

This is in particular true along the horizontal line $z = z_0 + h$, $h \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

$f'(z_0)$ is therefore a real number.
But z can also approach z_0 along the vertical line $z = z_0 + ih$, with $h \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{ih} = f'(z_0)$$

from which we conclude that $f'(z_0)$ is also purely imaginary, *and thus zero.*

We proved the following result: *If a real function of a complex variable is differentiable at a point, then its derivative is zero at this point.*

Example: The function $f(z) = |z|^2$ is not differentiable on $\mathbb{C} \setminus \{0\}$. It is differentiable at $z = 0$, and its derivative is then 0.

• For complex functions of complex variables, differentiability has fundamental consequences for the properties of the function. We now devote the next section (and many more in the remainder of this course) to this crucial case.

2 Analytic functions

2.1 Definition

Definition: A complex function f of a complex variable z is *analytic at* z_0 (or *holomorphic at* z_0) if

$$f'(w) = \lim_{z \to 0} \frac{f(z+w) - f(w)}{z}$$

exists for every w in an open neighborhood of z_0.

First consequence: if a function f is analytic at z_0, it is continuous at z_0. Indeed,

$$\lim_{z \to 0} [f(z_0 + z) - f(z_0)] = \lim_{z \to 0} z f'(z_0) = 0$$

Second consequence: If f and g are two functions that are analytic at z_0, then so is $f + g$, and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$
- So is also their product fg, and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
- So is also their quotient f/g provided $g(z_0) \neq 0$, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

- If f is analytic at z_0, and g is analytic at $w_0 = f(z_0)$, then $g \circ f$ is analytic at z_0, and $(g \circ f)'(z_0) = f'(z_0)g'(f(z_0))$

Examples:
It is easy to verify that $f(z) = z$ is analytic on $\mathbb{C}$, and that $g(z) = 1 + z^2$ is analytic on $\mathbb{C}$ and nonzero on $\mathbb{C} \setminus \{-i, i\}$. Hence $h(z) = z/(1 + z^2)$ is analytic on $\mathbb{C} \setminus \{-i, i\}$.

Any polynomial $P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$ is analytic on $\mathbb{C}$.
Any rational function $R(z) = P(z)/Q(z)$ with $P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$ and $Q(z) = d_m z^m + d_{m-1} z^{m-1} + \ldots + d_1 z + d_0$ polynomials, is analytic wherever it is defined, i.e. on all of $\mathbb{C}$ except for the points for which $Q(z) = 0$.

Definition: A complex function f of a complex variable z which is analytic for all $z \in \mathbb{C}$ is said to be an *entire* function.

2.2 Cauchy-Riemann equations

For any $z \in \mathbb{C}$, let us write $z = x + iy$, with $(x, y) \in \mathbb{R}^2$, and $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued functions.

If f is analytic at z_0, then $f'(z_0) = \lim_{z \to 0} (f(z_0 + z) - f(z_0))/z$ exists, independently of the path that z follows towards 0 in the complex plane (and consistent with the domain of definition of f). In particular,

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \left[\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h}\right] \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \qquad (2)
\end{aligned}$$

But we can also write

$$
\begin{aligned}
f'(z_0) &= \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} \frac{f(z_0+z)-f(z_0)}{z}\\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)}{ih}\\
&= -i\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0) \qquad (3)
\end{aligned}
$$

Equating (2) and (3), we must have

$$
\begin{cases}
\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}\\
\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}
\end{cases} \qquad (4)
$$

These are the well-known *Cauchy-Riemann equations*. **If a function f is holomorphic, then its real and imaginary parts satisfy the Cauchy-Riemann equations**.

Conversely, let us assume $f(z) = u(x,y) + iv(x,y)$ with u and v real valued functions which have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations and defined in some neighborhood of (x_0, y_0) with $z_0 = x_0 + iy_0$. Then, one may expand

$$
\begin{aligned}
u(x_0+h,y_0+k) &= u(x_0,y_0)+\frac{\partial u}{\partial x}h+\frac{\partial u}{\partial y}k+\epsilon_1 h+\epsilon_2 k\\
v(x_0+h,y_0+k) &= v(x_0,y_0)+\frac{\partial v}{\partial x}h+\frac{\partial v}{\partial y}k+\epsilon_3 h+\epsilon_4 k
\end{aligned} \qquad (5)
$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, $\epsilon_3 \to 0$, $\epsilon_4 \to 0$ as $(h,k) \to (0,0)$. Hence,

$$
\lim_{h+ik\to 0}\frac{f(z_0+h+ik)-f(z_0)}{h+ik} = \lim_{(h,k)\to(0,0)}\frac{\frac{\partial u}{\partial x}h+\frac{\partial u}{\partial y}k+i(\frac{\partial v}{\partial x}h+\frac{\partial v}{\partial y}k)}{h+ik}
$$

Using the Cauchy-Riemann equations, this becomes

$$
\lim_{(h,k)\to(0,0)}\frac{(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})(h+ik)}{h+ik} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}
$$

We conclude that f is analytic.

We have just proved the following results:

1) If $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z_0 = x_0 + iy_0$, then $u(x,y)$ and $v(x,y)$ have first-order partial derivatives which satisfy the Cauchy-Riemann equations

2) Conversely, if $f(z) = u(x,y) + iv(x,y)$, where u and v are real valued functions which have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations and are defined in some neighborhood of (x_0, y_0) with $z_0 = x_0 + iy_0$, then f is analytic at z_0.

We will prove later that the derivative of an analytic function is itself analytic. By this fact, the real part u and the imaginary part v of an analytic function have continuous partial derivatives of all orders. At that point, we will then have the following theorem:

Theorem: $f(z) = u(x,y) + iv(x,y)$ is analytic with continuous derivative $f'(a)$ at a iff $u(x,y)$ and $v(x,y)$ have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations.

The results above give us explicit ways to write $f'(z)$ in terms of the real and imaginary parts of f:

$$
f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}
$$

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic on all of $\mathbb{C}$ since

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and u_x, u_y, v_x, v_y are clearly continuous on $\mathbb{R}^2$.

$$f'(z) = 2x + 2iy = 2z$$

Polar coordinates

As we have seen in Lecture 1, it is often helpful to express a complex number in terms of polar coordinates, with $\Re(z) = r\cos\theta$ and $\Im(z) = r\sin\theta$. In that case, we have $f(z) = u(r,\theta) + iv(r,\theta) = f(r,\theta)$. Applying the appropriate change of variable to the Cauchy-Riemann equations, these equations take the following form in polar coordinates:

$$\begin{cases} r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{cases} \tag{6}$$

Example: Let $f(z) = 1/z^2$, for $z \neq 0$.

$$f(r,\theta) = \frac{\cos(2\theta) - i\sin(2\theta)}{r^2}$$

$$u_r = -2\frac{\cos(2\theta)}{r^3} \quad , \quad u_\theta = -2\frac{\sin(2\theta)}{r^2} \quad , \quad v_r = 2\frac{\sin(2\theta)}{r^3} \quad , \quad v_\theta = -2\frac{\cos(\theta)}{r^2}$$

We see that u_r, u_θ, v_r and v_θ satisfy the Cauchy-Riemann equations. These functions exist everywhere in the neighborhood of any point $(r,\theta) \neq (0,\theta)$, and are continuous at such a point. Therefore f is analytic on $\mathbb{C} \setminus \{0\}$, and for $z \neq 0$, $f'(z) = -2/z^3$.

2.3 Harmonic functions

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function. Then u and v satisfy the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Let us take for granted, for the time being, that u and v have continuous higher order partial derivatives (as discussed above, we will prove this later in this course). We can write

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

In other words, $\Delta u = 0$, $\Delta v = 0$. The real and imaginary parts of an analytic function satisfy Laplace's equation. They are *harmonic functions.*

If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then we say that v is a *conjugate harmonic function of* u.

Example: It is easy to see that $f(z) = z^3$ is analytic on $\mathbb{C}$. We can write $u(x,y) = x^3 - 3y^2x$, $v(x,y) = 3x^2y - y^3$, and compute

$$\Delta u = 6x - 6x = 0 \quad , \quad \Delta v = 6y - 6y = 0$$