1 Construction

1.1 Integrating a complex function over a curve in $\mathbb{C}$

• A natural way to construct the integral of a complex function over a curve in the complex plane is to link it to line integrals in $\mathbb{R}^2$ as already seen in vector calculus.

• We may understand this in two steps:

A) Consider a complex function $f(t) = u(t) + iv(t)$, for $t \in [a, b] \subset \mathbb{R}$, and u and v real valued functions. If f is a continuous function, we may define

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt \tag{1}$$

This definition, combined with the elementary properties of addition and multiplication in $\mathbb{C}$ we saw in Lecture 1, means that the integral has many intuitive properties that are reminiscent of the properties of integrals of real functions. Let us mention a few without proof, as these proofs are elementary:

- Let $c \in [a, b]$ and f continuous on $[a, b]$

$$\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt$$

$$\forall\lambda \in \mathbb{C}\,, \quad \int_a^b \lambda f(t)dt = \lambda\int_a^b f(t)dt$$

$$\Re\left(\int_a^b f(t)dt\right) = \int_a^b \Re(f(t))dt \;\;,\;\; \Im\left(\int_a^b f(t)dt\right) = \int_a^b \Im(f(t))dt$$

- Although the following property is also intuitive, let us prove that:

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt \tag{2}$$

If $\int_a^b f(t)dt = 0$, the inequality is trivial.
For $\int_a^b f(t)dt \neq 0$, let $\theta = \arg\left(\int_a^b f(t)dt\right)$

$$\left|\int_a^b f(t)dt\right| = \Re\left(e^{-i\theta}\int_a^b f(t)dt\right) = \Re\left(\int_a^b e^{-i\theta}f(t)dt\right) = \int_a^b \Re(e^{-i\theta}f(t))dt \le \int_a^b |f(t)|dt \quad \square$$

With this preliminary step in place, we are ready to define integration on a general curve in $\mathbb{C}$.

B) Let γ be a piecewise differentiable arc in the complex plane, with parametric equation

$$\gamma : \quad z = z(t)\;,\;\; a < t < b$$

If the function f is continuous on γ, then $f(z(t))$ is continuous on (a, b), and we define the integral of f on γ as the line integral

$$\int_\gamma f(z)dz := \int_a^b f(z(t))\frac{dz}{dt}dt \tag{3}$$

where the integral $\int_a^b$ may have to be split to match the intervals in which z is differentiable.

The definition above only makes sense if the integral is independent of the way the arc γ is parameterized. This is simple to check, using the rules for the change of variables for integrals of real valued functions. Imagine that another parameterization for γ is given by

$$\gamma \;:\; \tau \in (\alpha, \beta) \;\mapsto\; z(t(\tau))$$

with $t\ :\tau \in (\alpha,\beta)\ \mapsto t(\tau) \in (a,b)$ piecewise differentiable. Then,

$$\int_\gamma f(z)dz = \int_a^b f(z(t))\frac{dz}{dt}dt = \int_\alpha^\beta f(z(t(\tau)))\frac{dz}{dt}\frac{dt}{d\tau}d\tau$$
$$= \int_\alpha^\beta f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \qquad \square$$

1.2 Elementary properties

• Let $\gamma\ :\ z = z(t)\ ,\ t \in (a,b)$. We define the opposite arc, written $-\gamma$, by

$$-\gamma\ :\ z = z(-t)\ ,\ t \in (-b,-a)$$

Then,

$$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f(z(-t))\frac{d}{dt}\left[z(-t)\right]dt = -\int_{-b}^{-a} f(z(-t))\frac{dz}{dt}(-t)dt = -\int_a^b f(z(t))\frac{dz}{dt}(t)dt$$

where the last equality is obtained with a simple change of variable. Hence

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz \tag{4}$$

• Linearity as an operator on functions

Let f and g be two continuous functions on the piecewise differentiable arc γ, and $(\alpha,\beta) \in \mathbb{C}^2$

$$\int_\gamma (\alpha f + \beta g)dz = \alpha\int_\gamma f dz + \beta\int_\gamma g dz \tag{5}$$

• Linearity as an operator on curves

Consider an arc γ which can be subdivided into two piecewise-differentiable arcs γ_1 and γ_2, and f a continuous function on γ. Then

$$\int_\gamma f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz = \int_{\gamma_2} f dz + \int_{\gamma_1} f dz \tag{6}$$

We can use this property to show that an integral over a closed curve does not depend on the starting point on the curve. Indeed, consider two such points P and Q, corresponding to different parameterizations, as shown in Figure 1. If we call γ_1 the part of γ from P to Q, and γ_2 the part of γ from Q to P,

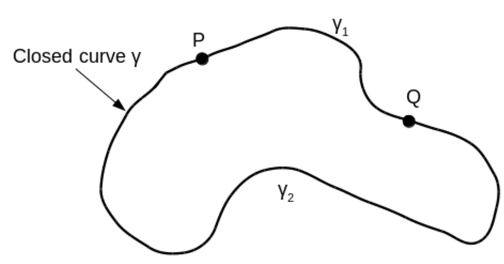

Figure 1: Closed curve γ subdivided into the arcs γ_1 and γ_2

$$\int_\gamma f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz = \int_{\gamma_2} f dz + \int_{\gamma_1} f dz$$

The expression in the middle corresponds to the evaluation of the integral starting from the point P, while the expression on the right corresponds to the evaluation of the integral starting from the point Q.

1.3 Examples

• Let us start with a very simple example which will play a fundamental role in the rest of this course.

Let $a \in \mathbb{C}$, and consider the integral

$$\int_\gamma \frac{dz}{z-a}$$

where γ is the closed circle with radius R and centered in a. A simple parameterization for γ is $\gamma:\ \theta \in [0, 2\pi) \ \mapsto\ z(\theta) = Re^{i\theta} + a$. Thus

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} i d\theta = 2\pi i$$

• Let us compute

$$\int_\gamma z^2 dz$$

where γ is the line segment between 0 and $2+i$.

A simple parameterization for γ is

$$\gamma\ :\ z(t) = (2+i)t \ \ ,\ \ 0 \le t \le 1$$

We then have

$$\int_\gamma z^2 dz = \int_0^1 (2+i)^3 t^2 dt = (2+11i)\int_0^1 t^2 dt = \frac{2}{3} + \frac{11}{3}i$$

2 The fundamental theorem of calculus for integrals in $\mathbb{C}$

2.1 Line integrals with respect to x and y

The line integral with respect to $\overline{z}$ is defined as

$$\int_\gamma f(z) d\overline{z} := \overline{\int_\gamma \overline{f(z)} dz} \tag{7}$$

Line integrals with respect to $x = \Re(z)$ and $y = \Im(z)$ along the arc γ are then naturally constructed as

$$\int_\gamma f(z)dx = \frac{1}{2}\left(\int_\gamma f(z)dz + \int_\gamma f(z)d\overline{z}\right) \ \ ,\ \ \int_\gamma f(z)dy = \frac{1}{2i}\left(\int_\gamma f(z)dz - \int_\gamma f(z)d\overline{z}\right) \tag{8}$$

If we then write $f(z) = u(x,y) + iv(x,y)$, with $z = x+iy$, we have

$$\int_\gamma f(z)dz = \int_\gamma f(z)dx + i\int_\gamma f(z)dy = \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx) \tag{9}$$

which can be viewed as another definition for $\int_\gamma f(z)dz$, involving only line integrals of scalar functions, as already introduced in vector calculus.

2.2 Independence of path

We have just reduced the complex integral $\int_\gamma f(z)dz$ to line integrals of the form $\int_\gamma P(x,y)dx + Q(x,y)dy$. We will now recall a well-known result of vector calculus on independence of path to determine when $\int_\gamma f(z)dz$ only depends on the endpoints of γ and not the actual path γ describes.

Theorem: Let Ω be an open connected set of $\mathbb{R}^2$, and P and Q two functions that are continuous on Ω, and potentially complex valued. The integral $\int_\gamma Pdx + Qdy$ depends only on the end points of γ iff there exists a function $U(x,y)$ on Ω with the partial derivatives $P(x,y) = \frac{\partial U}{\partial x}$, $Q(x,y) = \frac{\partial U}{\partial y}$.

Proof: The sufficient condition is straightforward: if such a U exists, then

$$\int_\gamma Pdx + Qdy = \int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}\right) = \int_a^b \frac{d}{dt}\Big[U(x(t),y(t))\Big]dt = U(x(b),y(b)) - U(x(a),y(a))$$

for any arc γ between the points $(x(a),y(a))$ and $(x(b),y(b))$.

- Conversely, if $\int_\gamma P(x,y)dx + Q(x,y)dy$ only depends on the end points, we can construct a single valued function U by fixing a point $(x_0,y_0) \in \Omega$, and defining

$$U(x,y) = \int_\gamma P(x,y)dx + Q(x,y)dy$$

where γ is any arc between (x_0,y_0) and (x,y). We now show that U satisfies the conditions of the theorem.

Consider the point $(x+\Delta x, y)$, and any arc γ' between (x_0,y_0) and $(x+\Delta x,y)$. For Δx sufficiently small, there exists an arc γ'' in Ω between (x,y) and $(x+\Delta x,y)$, and parallel to the x-axis. By the independence of path of the integral, we can write

$$\begin{aligned} U(x+\Delta x,y) &= \int_{\gamma'} P(x,y)dx + Q(x,y)dy \\ &= \int_\gamma P(x,y)dx + Q(x,y)dy + \int_{\gamma''} P(x,y)dx \\ &= U(x,y) + \int_{\gamma''} P(x,y)dx \end{aligned}$$

Constructing arcs γ'' in this manner for all small Δx, we may write

$$\lim_{\Delta x\to 0}\frac{U(x+\Delta x,y)-U(x,y)}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_x^{x+\Delta x} P(x,y)dx = P(x,y)$$

where the last equation follows from the continuity of P. We thus have $\frac{\partial U}{\partial x} = P(x,y)$.
With a very similar proof, we would show that $\frac{\partial U}{\partial y} = Q(x,y)$, which concludes our proof. □

2.3 The fundamental theorem of calculus for integrals in $\mathbb{C}$

Consider $f(z) = u(x,y) + iv(x,y)$, $P(x,y) = u(x,y) + iv(x,y)$, and $Q(x,y) = i(u(x,y) + iv(x,y))$. Then

$$\int_\gamma f(z)dz = \int_\gamma P(x,y)dx + Q(x,y)dy$$

The integral on the right-hand side depends on the end points if and only if there exists $F(x,y)$ such that $P(x,y) = \frac{\partial F}{\partial x}$, and $Q(x,y) = \frac{\partial F}{\partial y}$. If such an F exists, then

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

Writing $F(z) = U(x,y) + iV(x,y)$, the equality above becomes Cauchy-Riemann equations for U and V. So F is analytic, with derivative f. We have proven the following theorem.

Theorem (Fundamental theorem of calculus for integrals in $\mathbb{C}$): The integral $\int_\gamma f(z)dz$, with f continuous on an open connected set Ω containing γ, depends only on the end points of γ iff f is the derivative of an analytic function F in Ω.
We say that F is a primitive of f.

Corollary: if $f(z) = \frac{dF}{dz}$ where F is analytic on an open connected set Ω and if γ is a closed curve in Ω, then

$$\oint_\gamma f(z)dz = 0 \tag{10}$$

Conversely, if f is a continuous function on an open connected set Ω and is such that $\oint_\gamma f(z)dz = 0$ for any closed contour in Ω, then f has a primitive.

The proof of the latter is left to the reader as an enlightening exercise, close to what we have done previously in this lecture.

Example 1:

$$\forall z \in \mathbb{C}, \frac{d}{dz}\left(\frac{z^3}{3}\right) = z^2$$

Thus, if γ is the straight line from 0 to $2+i$, we have

$$\int_\gamma z^2 dz = \frac{1}{3}(2+i)^3 - \frac{1}{3}(0)^3 = \frac{2}{3} + \frac{11}{3}i$$

which is the result we found previously with the direct method.

Example 2: • Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$

$$(z-a)^n = \frac{d}{dz}\left[\frac{(z-a)^{n+1}}{n+1}\right]$$

and $(z-a)^{n+1}/(n+1)$ is entire, so $\int_\gamma (z-a)^n dz = 0$ for all closed curves γ in $\mathbb{C}$.

• For $n = -1$, we have already seen that the result does not hold

• For $n = -k$, $k \in \mathbb{N}^* \setminus \{1\}$, the result holds for any curve γ in $\mathbb{C}$ that does not go through a

3 Integration with respect to arc length

We will often encounter integrals with respect to arc length, defined by

$$\int_\gamma f(z)ds := \int_\gamma f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt \tag{11}$$

As before, this only makes sense if the integral is independent of the parameterization. This can be verified easily, as well as the fact that

$$\int_{-\gamma} f(z)ds = \int_\gamma f(z)ds$$

The length of a curve γ in the complex plane is given by

$$L(\gamma) = \int_\gamma ds = \int_\gamma |dz|$$

Finally, using Eq.(2) we have the triangle inequality:

$$|\int_\gamma f(z)dz| \leq \int_\gamma |f(z)||dz| \leq L(\gamma)\sup_{z\in\gamma}|f(z)| \tag{12}$$

4 The Cauchy-Goursat theorem

A central result of complex analysis and of this lecture is the following theorem:

Theorem: Let f be an analytic function in the open connected set Δ' obtained by omitting a finite number of points ζ_i from an open disk Δ. If f satisfies the condition $\lim_{z\to\zeta_i}(z-\zeta_i)f(z) = 0$ for all i, then $\int_\gamma f(z)dz = 0$ for any closed rectifiable arc γ in Δ'.

We will prove this theorem, called *the Cauchy-Goursat theorem*, step by step, starting with a very simple domain. Let us first begin with a short discussion of rectifiable arcs.

4.1 Rectifiable arcs

Consider the arc $\gamma : z = z(t)$, $a \leq t \leq b$. We have seen a way to define its length if it is piecewise differentiable. A more general definition is given by the least upper bound of all sums of the form

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \ldots + |z(t_n) - z(t_{n-1})|$$

with $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$.
If this least upper bound is finite, γ is said to be a rectifiable arc. Any piecewise differentiable arc is rectifiable, and in that case the two definitions of length are equivalent.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_\gamma f(z)dz = \lim_{\substack{N\to\infty, \\ |z(t_k)-z(t_{k-1})|\to 0}} \sum_{k=1}^{N} f(z(t_k))(z(t_k) - z(t_{k-1}))$$

In this course, we will never have to consider arcs which are not piecewise differentiable, but it is important to know that many of the theorems hold with weaker assumptions on γ.

4.2 The Cauchy-Goursat Theorem for a rectangle

Theorem: Let R be the rectangle in the complex plane given by $a \leq x \leq b$, $c \leq y \leq d$, with $x = \Re(z)$ and $y = \Im(z)$, and ∂R is boundary curve, i.e. the arc following the boundary of R in the counterclockwise direction.
If a function f is analytic on an open set which contains R, then $\int_{\partial R} f(z)dz = 0$.

The proof we give here is at once elegant and simple. It was first found by the French mathematician Edouard Goursat. The proof starts by bisecting R into four congruent rectangles R_1, R_2, R_3, and R_4, as shown in Figure 2, and looking for an upper bound for $\int_{\partial R} f(z)dz$ in terms of an integral on one of the smaller rectangles.

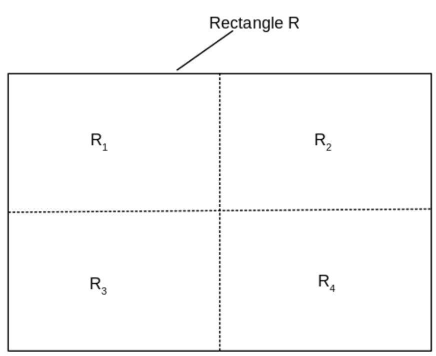

Figure 2: Bisection of the rectangle R into 4 congruent rectangles

For any rectangle $\tilde{R}$ inside R, R included, we define the number

$$\eta(\tilde{R}) = \int_{\partial\tilde{R}} f(z)dz$$

We can write

$$\eta(R) = \eta(R_1) + \eta(R_2) + \eta(R_3) + \eta(R_4)$$

since the integrals along shared sides cancel. Hence $\exists i$ such that $|\eta(R)| \leq 4|\eta(R_i)|$. The key idea is to repeat the bisection process in that R_i, to bound $\eta(R_i)$. After n subdivisions of this type, we can write

$$|\eta(R^{(n)})| \geq \frac{1}{4}|\eta(R^{(n-1)})| \geq \ldots \geq \frac{1}{4^n}|\eta(R)|$$

where $\eta(R^{(j)})$ is the rectangle used for the upper bound at the j^{th} subdivision.

Now, $\forall\delta > 0$, $\exists N \in \mathbb{N}$ and $z_0 \in R$ such that

$$\forall n \geq N \ , \ R^{(n)} \subset \{|z - z_0| < \delta \ , \ z \in \mathbb{C}\} \tag{13}$$

Furthermore, since f is analytic in R, $\forall\epsilon > 0$, $\exists\delta$ such that

$$|z - z_0| < \delta \ \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

For this δ, it is possible to subdivide R enough times such that $R^{(n)}$ satisfies Eq. (13). We can then write, using previous results from these notes,

$$\eta(R^{(n)}) = \int_{\partial R^{(n)}} f(z)dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]dz$$

so that

$$|\eta(R^n)| \leq \epsilon \int_{\partial R^{(n)}} |z - z_0||dz|$$

Now, for z on $\partial R^{(n)}$, $|z - z_0| \leq d_n$, where d_n is the length of the diagonal of $R^{(n)}$. Thus, if L_n is the length of its perimeter,

$$|\eta(R^n)| \leq \epsilon d_n L_n$$

Finally, if d is the length of the diagonals of R and L the length of its perimeter,

$$d_n = \frac{d}{2^n} \quad , \quad L_n = \frac{L}{2^n}$$

We conclude that

$$|\eta(R)| \leq 4^n |\eta(R^{(n)})| \leq \epsilon dL$$

Since ϵ is arbitrarily small, $\eta(R) = 0$ $\square$

Our goal is to now generalize this result for cases in which f may not be analytic at a finite number of points ϵ_i inside R:

Theorem: Let f be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_i. If $\forall i$, $\lim_{z\to\zeta_i} (z - \zeta_i)f(z) = 0$, then

$$\int_{\partial R} f(z)dz = 0$$

Proof: Without loss of generality, we assume that f is not analytic at only one point ζ in R. We then subdivide R as shown in Figure 3, where S_0 is a square with center ζ.

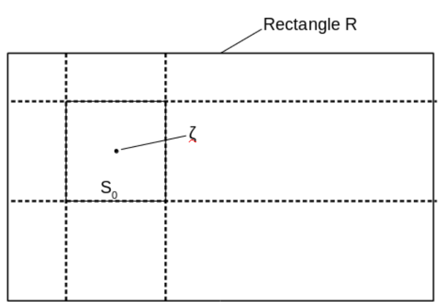

Figure 3: Subdivision of the rectangle R around the point ζ where f is not analytic

Using the previous theorem,

$$\int_{\partial R} f(z)dz = \int_{\partial S_0} f(z)dz$$

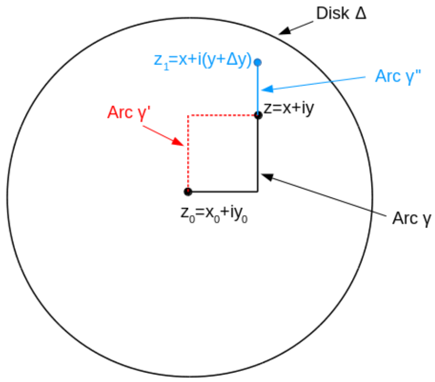

Figure 4: Disk Δ and relevant arcs for the first part of the proof of Cauchy's theorem

Now, $\forall \epsilon > 0$, we may choose S_0 small enough that

$$\forall z \in \partial S_0 \quad , \quad |f(z)| \leq \frac{\epsilon}{|z-\zeta|}$$

Hence,

$$\left| \int_{\partial R} f(z)dz \right| \leq \epsilon \int_{\partial S_0} \frac{|dz|}{|z-\zeta|} \leq \epsilon \frac{4l}{\frac{l}{2}} = 8\epsilon$$

where l is the length of a side of the square. Since ϵ is arbitrarily small, $\int_{\partial R} f(z)dz = 0$ □

4.3 The Cauchy-Goursat theorem for a disk

Theorem: If f is analytic in an open disk Δ, then $\int_\gamma f(z)dz = 0$ for every closed curve γ in Δ.

Proof: The proof is very similar in spirit to our proof for the independence of path above, but also uses Cauchy's theorem for a rectangle.

Consider the disk Δ centered in $z_0 = x_0 + iy_0$, and the point $z = x + iy$, and γ the arc that is horizontal from (x_0, y_0) to (x, y_0), and vertical from (x, y_0) to (x, y), as shown in Figure 4.

We define

$$F(z) = \int_\gamma f(z)dz$$

We have

$$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{F(x, y+\Delta y) - F(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma''} f(z)dz = if(z)$$

where γ'' is the vertical line from (x, y) to $(x, y + \Delta y)$ (see Figure 4).

Now, by Cauchy's theorem on rectangles, one can also write

$$F(z) = \int_{\gamma'} f(z)dz$$

where γ' is shown in Figure 4. Thus, applying the same reasoning, we can also find

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x, y) - F(x,y)}{\Delta y} = f(z)$$

We conclude that $\partial F/\partial x = -i\partial F/\partial y$. We can thus say as before that F is analytic, so that by the fundamental theorem of calculus

$$\int_\gamma f(z)dz = 0$$

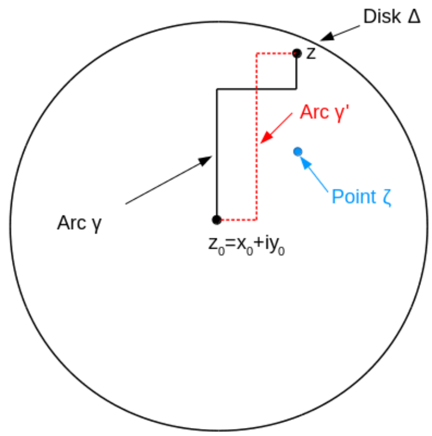

Figure 5: Disk Δ and relevant arcs for the second part of the proof of Cauchy's theorem

for any closed curve γ in Δ $\square$

We are now ready to prove Cauchy's theorem in its full extent, as stated at the beginning of these notes.

Proof: Without loss of generality, we can again assume that there is only one special point ζ in Δ. We define $F(z) = \int_\gamma f(z)dz$ in a similar way as before; we just have to be careful with the location of ζ with respect to the arcs we used in the proof.

First case: ζ lies neither on the line $x = x_0$ nor on the line $y = y_0$, where $z_0 = x_0 + iy_0$ is the center of Δ and the initial point of γ. Then it is possible to construct a path γ from z_0 to any $z \neq \zeta$ made only of horizontal and vertical line segments (three segments may be needed) with the last segment a vertical segment and where γ does not go through ζ. This can be seen in Figure 5.
It is then easy to show, in the same way as before, that $F_y(z) = if(z)$.

We know by Cauchy's theorem on a rectangle that $F(z) = \int_{\gamma'} f(z)dz$, with γ' shown in Figure 5, and that therefore $F_x(z) = f(z)$. We conclude that F is analytic, so $\int_\gamma f(z)dz = 0$ for any closed curve in Δ'.

Second case: ζ lies on the line $x = x_0$ or on the line $y = y_0$. In that case, one just moves the starting point z_0 for the definition of F away from $x_0 + iy_0$ to return to the first case $\square$

4.4 Looking ahead: the Cauchy-Goursat theorem for more general regions

In the next couple of lectures, we will be able to derive many results only using the versions of the Cauchy-Goursat theorems discussed above. However, you may naturally wonder whether this key theorem holds for more general domains. The answer is yes, with different forms depending on the regions.

Simply connected domains

Theorem: If a function f is analytic throughout a simply connected domain Ω, then

$$\int_C f(z)dz = 0$$

for every closed contour C lying in Ω.

We will not prove this important result here, but in the class textbook you can find a proof based on many of the ideas we used for the proofs above.

Multiply connected domains

For multiply connected domains, the situation is more complicated, as one can sense from the result

$$\int_{C_R(a)} \frac{dz}{z-a} = 2\pi i$$

where $C_R(a)$ is the circle with radius R and center a.

In the next few lectures, we will learn tools which will allow us to state the most general form of the Cauchy-Goursat theorem, which is applicable to both multiply connected and simply connected domains.