1 Cauchy's Integral Formula

1.1 Index of a point with respect to a closed curve

Let $z \in \mathbb{C}$, and a piecewise differentiable closed curve γ which does not pass through z. The value of the integral

$$\int_\gamma \frac{d\zeta}{\zeta - z}$$

is a multiple of $2\pi i$.

Indeed, let $\gamma : \zeta = \zeta(t)\ ,\ a \leq t \leq b$, and consider the function

$$f(t) = \int_a^t \frac{1}{\zeta(u) - z} \frac{d\zeta}{du} du$$

Since γ does not pass through z, f is defined and continuous on $[a, b]$. Furthermore, for all t such that $\frac{d\zeta}{dt}(t)$ is continuous, we can write

$$f'(t) = \frac{1}{\zeta(t) - z} \frac{d\zeta}{dt} \qquad \Leftrightarrow \qquad \frac{d}{dt}\left[e^{-f(t)}(\zeta(t) - z)\right] = 0$$

Let us call $g(t) := e^{-f(t)}(\zeta(t) - z)$. Since γ is piecewise differentiable and since g is continuous on γ, we have

$$g(t) = Cst = g(a)$$

from which we conclude that

$$e^{f(t)} = \frac{\zeta(t) - z}{\zeta(a) - z}$$

For a closed curve γ, $\zeta(b) = \zeta(a)$, so

$$e^{f(b)} = e^{f(a)} = 1 \qquad \Leftrightarrow \quad \exists k \in \mathbb{Z} \ \ s.t. \ \ f(b) = 2\pi k i$$

Example

In homework 4, you showed that if $\mathcal{C}$ is the unit circle centered in 0 and traversed in the counterclockiwise direction,

$$\int_\mathcal{C} \frac{dz}{z - \frac{1}{2}} = 2\pi i$$

Definition: The *index of the point z with respect to the closed curve γ* is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \tag{1}$$

n can be viewed as a quantity measuring the number of times a closed curve winds around a fixed point not on it. For this reason, n is often called the *winding number*.

Theorem: Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each open connected set of $\mathbb{C} \setminus \{\gamma\}$, and zero if this set is unbounded.

Proof: The function

$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

is integer valued on any open connected set of $\mathbb{C} \setminus \{\gamma\}$, and continuous on these sets. Since the image $f(\Omega)$ of any such set Ω is also connected, and the only connected subsets of the integers contain at most one point, f is constant.

In addition, for $|z|$ sufficiently large, there is a disk of radius R such that γ is contained in the disk but z is not. Then a direct application of Cauchy's theorem tells us that $n(\gamma, z) = 0$. This result then holds for the entire region by continuity.

1.2 Cauchy's integral formula

Theorem: Suppose that f is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point z not on γ

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \tag{2}$$

where $n(\gamma, z)$ is the index of z with respect to γ.

Proof: Let Δ be an open disk, γ a closed curve in Δ, and $z \in \Delta$ which does not lie on γ. We consider the function

$$F : \zeta \in \Delta \setminus \{z\} \mapsto \frac{f(\zeta) - f(z)}{\zeta - z}$$

From the hypotheses of the theorem, we know that F is analytic on $\Delta \setminus \{z\}$, and that

$$\lim_{\zeta \to z} F(\zeta)(\zeta - z) = 0$$

Hence, by Cauchy's theorem we know that $\int_\gamma F(\zeta)d\zeta = 0$, i.e

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = f(z)\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i n(\gamma, z)f(z) \quad \square$$

Note that the properties of the function in the theorem can be relaxed to a function which is analytic in Δ except at a finite number of points ξ_i, provided that $\forall i, \lim_{z \to \xi_i} (z - \xi_i)f(z) = 0$. Cauchy's integral formula still holds in that case. The proof is left for the reader.

Examples

• Let $\mathcal{C}$ be the unit circle centered in 0 and traversed in the counterclockwise direction.

$$\int_\mathcal{C} \frac{\cos z}{z}dz = 2\pi i n(\mathcal{C}, 0)\cos(0) = 2\pi i$$

• Let γ be the arc composed of the line segment $[-2\sqrt{3}\ ,\ 2\sqrt{3}]$ along the real axis, and the upper half of the ellipse in $\mathbb{R}^2$ given by the Cartesian equation $x^2/12 + y^2/25 = 1$, traversed once in the counterclockwise direction. We have

$$\int_\gamma \frac{e^{iz}dz}{1 + z^2} = \frac{1}{2i}\left(\int_\gamma \frac{e^{iz}dz}{z - i} - \int_\gamma \frac{e^{iz}dz}{z + i}\right) = \pi(e^{-1}n(\gamma, i) - en(\gamma, -i)) = \frac{\pi}{e}$$

Cauchy's formula gives an expression for $f(z)$ only knowing that f is analytic in Δ and knowing the values of f on γ. This will be useful to prove many key theorems, and to study the local properties of functions. Here is a direct illustration:

Theorem (The mean value property for analytic functions): The value of an analytic function f at z is equal to the average of its values around any circle $|\zeta - z| = R$ inside the domain where it is analytic.

Proof: The result comes directly from Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(z + Re^{i\theta})d\theta$$

You probably came across a similar theorem for harmonic functions of real variables. The connection is clear, through the Cauchy-Riemann equations.

1.3 Derivatives of f

It is tempting to differentiate Cauchy's formula under the integral sign to obtain analogous formulae for the derivatives of f. To do so, we need a short lemma regarding that operation:

Lemma: Consider an open connected set Ω of $\mathbb{C}$, and γ an arc in Ω. If φ is continunous on γ, then

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$$

is analytic in $\Omega \setminus \{\gamma\}$, and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Proof: We prove this lemma by induction.

• The lemma is true for $n = 0$.

• Let us assume that it holds for $n-1$: F_{n-1} is analytic on $\Omega \setminus \{\gamma\}$ for any φ continuous on γ, and $F_{n-1}'(z) = (n-1)F_n(z)\ \forall z \in \Omega \setminus \{\gamma\}$

• Let $z_0 \in \Omega \setminus \{\gamma\}$, and consider a neighborhood $D_\delta(z_0)$ that does not meet γ, and inside that neighborhood a smaller neighborhood $D_{\delta/2}(z_0)$. Observe that

$$z \in D_{\delta/2}(z_0)\ \Rightarrow \begin{cases} |z - z_0| < \frac{\delta}{2} \\ |\zeta - z| > \frac{\delta}{2}\ ,\ \forall \zeta \in \gamma \end{cases}$$

For any continuous function φ on γ, we may write

$$\begin{aligned} F_n(z) - F_n(z_0) &= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta = \int_\gamma \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta \\ &= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta + (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta \end{aligned}$$

Let us define $\psi(\zeta) := \varphi(\zeta)/(\zeta - z_0)$, which is continuous on γ. We can rewrite the equality above as

$$F_n(z) - F_n(z_0) = \left[\int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^{n-1}} - \int_\gamma \frac{\psi(\zeta)}{(\zeta - z_0)^{n-1}} \right] + (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta \tag{3}$$

Now, $\forall z \in D_{\delta/2}(z_0)$,

$$\left| (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta \right| \leq |z - z_0| \left(\frac{2}{\delta} \right)^n \int_\gamma |\psi(\zeta)||d\zeta|$$

so

$$\lim_{z \to z_0} (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta = 0$$

since ψ is continuous on γ and γ is rectifiable. Furthermore, we know by the induction hypothesis that the term in brackets in Eq. (3) goes to zero as $z \to z_0$. Hence, for any φ continuous on γ, F_n is continuous in z_0.

Defining

$$G_n(z) := \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta$$

we may write

$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + G_n(z)$$

By the induction hypothesis, the first term on the right goes to $G_{n-1}'(z_0) = (n-1)G_n(z_0)$ as $z \to z_0$, and from our previous point we also know that G_n is continuous, so we find

$$\lim_{z \to z_0} \frac{F_n(z) - F_n(z_0)}{z - z_0} = (n-1)G_n(z_0) + G_n(z_0) = nG_n(z_0) = nF_{n+1}(z_0) \quad \square$$

The lemma gives us the following important result:

Let f be a function which is analytic in an open connected set Ω. For any point z_0 in Ω, we consider a neighborhood $D_\delta(z_0) \subset \Omega$, and a circle C with center z_0 inside $D_\delta(z_0)$. For all points in the interior of C, we can use Cauchy's integral formula to write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Applying the lemma, we can say that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \tag{4}$$

is analytic in the interior of C. More generally,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{5}$$

is analytic in the interior of C. We have therefore proven the following central result of complex analysis:

An analytic function on the open connected set Ω has derivatives of all orders in Ω, which are themselves analytic.

2 Consequences of Cauchy's integral formula

2.1 Morera's theorem

Theorem: If f is defined and continuous in an open connected set Ω and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in Ω, then f is analytic in Ω.

Proof: From Lecture 4, we know that given the hypotheses of the theorem, f has a primitive in Ω. By the result we just found, f, the derivative of an analytic function in Ω, is analytic itself.

2.2 Cauchy's estimate

Suppose f is analytic in a disk $|z - z_0| \leq R$, and bounded on the circle γ given by $|z - z_0| = R$: $\forall\ z \in \gamma\ ,\ |f(z)| \leq M$ with $M \in \mathbb{R}_+$. Then

$$\begin{aligned} |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \int_\gamma \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| |d\zeta| \\ &\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R \end{aligned}$$

So we conclude that

$$|f^{(n)}(z_0)| \leq n! \frac{M}{R^n} \tag{6}$$

This inequality is known as Cauchy's estimate. It can be used for the well-known Liouville theorem below.

2.3 Liouville's theorem

Theorem: A bounded entire function is constant.

Proof: Let M be this bound. Then, using Cauchy's estimate, we have that

$$\forall\ z \in \mathbb{C}\ ,\ ,\forall R > 0, |f'(z)| \leq \frac{M}{R}$$

Hence $f'(z) = 0$, which means that f is constant.

2.4 The fundamental theorem of algebra

Theorem: Every polynomial of degree $n \geq 1$ has n roots.

Proof: Assume that $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ does *not* have a root. Then $g(z) := 1/P(z)$ is an entire function. Furthermore, g is bounded since

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|z|^n} = |a_n| \quad \Rightarrow \quad \lim_{|z|\to\infty} \frac{1}{P(z)} = 0$$

By Liouville's theorem, $1/P(z)$ must be a constant equal to zero, which is not possible. Hence, P has at least one root α, and we can write

$$P(z) = (z - \alpha)Q(z)$$

Repeating the steps for Q, we find that P must eventually have n roots. $\square$

2.5 Power series

Theorem: If f is analytic in an open connected set Ω which contains a closed disk $\overline{D_R(z_0)}$, then f has a power series expansion at z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

which is convergent for all $z \in D_R(z_0)$, with

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof: $\forall\ z \in D_R(z_0)$, $\forall \zeta \in C_R(z_0)$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} (\zeta - z_0)^{-n-1} (z - z_0)^n$$

Since convergence is uniform in $\zeta \in C_R(z_0)$, we can use Cauchy's formula to write

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \oint_{C_R(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta) \sum_{n=0}^{\infty} (\zeta - z_0)^{-n-1} (z - z_0)^n d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta) (\zeta - z_0)^{-n-1} d\zeta \right) (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \ \square \end{aligned}$$

2.6 Maximum modulus principle

Another key consequence of the Cauchy integral formula and its associated mean value property for analytic functions is that the maximum modulus principle, which may be expressed as follows. If f is analytic and nonconstant in an open connected set Ω, then its modulus $|f|$ has no maximum in Ω. We prove this result in two steps.

Lemma: Suppose f is analytic in a neighborhood $|z - z_0| < \epsilon$ of a point $z_0 \in \mathbb{C}$, and that $|f(z)| \leq |f(z_0)|$ at each point z of this neighborhood. Then f has the constant value $f(z_0)$ throughout that neighborhood.

Proof: Let us assume that f satisfies the hypotheses of the lemma. We apply the mean value property for f on the circle $C_\delta(z_0)$ oriented in the counterclockwise direction, where δ is chosen small enough that the circle is inside the ϵ-neighborhood of the theorem:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \delta e^{i\theta}) d\theta$$

which implies

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\delta e^{i\theta})| d\theta$$

On the other hand, by hypothesis $\forall\theta \in [0,2\pi]$, $|f(z_0+\delta e^{i\theta})| \leq |f(z_0)|$, so

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\delta e^{i\theta})| d\theta \leq \frac{|f(z_0)|}{2\pi}\int_0^{2\pi} d\theta = |f(z_0)|$$

We conclude that

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0+\delta e^{i\theta})|\right] d\theta = 0$$

The integrand in the integral above is a continuous function of θ which is always positive by hypothesis. Hence, $\forall z \in C_\delta(z_0)$, $|f(z)| = |f(z_0)|$.

Now, δ was chosen arbitrarily; the only condition is that δ must be chosen small enough for $C_\delta(z_0)$ to be in the ϵ-neighborhood. We conclude that for all z in the ϵ-neighborhood of z_0, $|f(z)| = |f(z_0)|$.
f is an analytic function whose modulus is constant in the ϵ-neighborhood $z \in \mathbb{C} : |z - z_0| < \epsilon$. By a direct application of the Cauchy-Riemann equations, this means that f itself is constant in that ϵ-neighborhood □

We are now ready to state and prove the maximum modulus principle.

Theorem: If a function f is analytic and not constant in an open connected set Ω, then its modulus $|f|$ has no maximum in Ω.

Proof: To prove the theorem, we will consider a holomorphic function f, and assume that f does have a maximum z_0 in Ω. We will show that this implies that f is constant in Ω.

The lemma already proves the result for the case when Ω is an open disk. We just need to extend the proof to a general open connected set Ω. This is done as follows.

Consider the point z_0 where f has a global maximum in Ω, and any other point $z \in \Omega$. We connect z_0 and z with a polynomial line L such that L is always in the interior of Ω. Let δ be the shortest distance from points on L to the boundary of Ω. There exists a finite sequence of points $z_0, z_1, \ldots, z_{n-1}, z_n = z$ on L such that

$$\forall\ k \in [\![1, n]\!]\ \ ,\ |z_k - z_{k-1}| < \delta$$

We can then form a sequence of neighborhoods $N_0, N_1, \ldots, N_{n-1}, N_n$ where each N_k has z_k as its center, and radius δ. This construction is sketched in Figure **??**.

By construction, each of these neighborhoods is in the interior of Ω, and f is analytic on all of them. Furthermore, the center of each neighborhood N_k lies in the neighborhood N_{k-1}.

Now, $|f|$ has a global maximum at z_0, which is therefore also the maximum of $|f|$ in N_0. By the lemma we just proved,

$$\forall z \in N_0\ ,\ f(z) = f(z_0)$$

In particular, $f(z_1) = f(z_0)$.

Now, since $|f(z_0)|$ is a global maximum in Ω, $|f(z_1)| = |f(z_0)|$ is a global maximum in N_1, and we can write

$$\forall z \in N_1\ ,\ |f(z)| \leq |f(z_1)|$$

Applying the lemma again,

$$\forall z \in N_1\ ,\ f(z) = f(z_1) = f(z_0)$$

and in particular $f(z_2) = f(z_1) = f(z_0)$.

Repeating the process through the n successive neighborhoods, we find

$$f(z_n) = f(z_{n-1}) = \ldots = f(z_1) = f(z_0)$$

Finally, remember that $z_n = z$ was chosen arbitrarily in Ω. Thus, we proved that $\forall z \in \Omega$, $f(z) = f(z_0)$. f is a constant function. This concludes our proof □

We note finally that the maximum modulus principle is often formulated in the following equivalent manner:

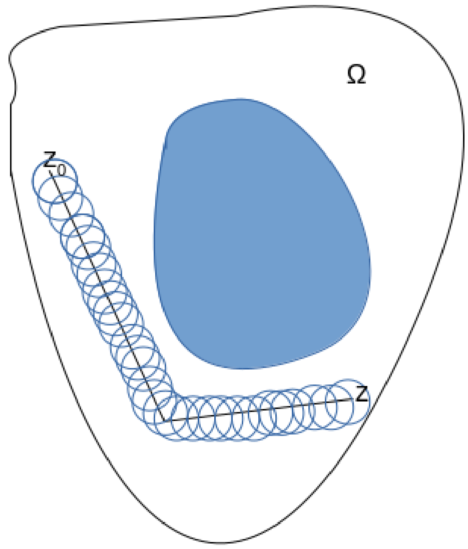

Figure 1: Construction of the string of overlapping neighborhoods from z_0 to z for the proof of the maximum modulus principle. For generality, we considered an open set Ω which is connected but not simply connected: the blue region in the center does not belong to Ω.

If f is defined and continuous on a closed bounded set E, and analytic in the interior of E, then the maximum of $|f|$ on E is assumed on the boundary of E.

Illustration: remember problem 3 in HW #2, stating that there does not exist analytic functions of $z = x + iy$ whose modulus is equal to $K/\cosh x$, with $K \neq 0$ constant.

The maximum modulus principle gives us a quick proof of this for functions which are analytic on an open connected set which contains a subset of the imaginary axis in its interior: $\forall y \in \mathbb{R}$, $z = 0 + iy$ is a maximum of $|f|$, since $\cosh 0$ is a minimum of cosh. Since f cannot be constant by hypothesis, this would contradict the maximum modulus principle, so no such f exists.