1 Taylor series

1.1 Taylor series for analytic functions

We start this lecture by summarizing in one place several important results we have obtained in previous lectures. We will omit the proofs, which were already given in these lectures.

Theorem (Taylor series): If f is analytic in an open connected set Ω which contains a closed disk $\overline{D_R(z_0)}$, then f has a *Taylor series* at z_0, written

$$f(z) = \sum_{n=0}^{+\infty} c_n(z-z_0)^n$$

which is absolutely convergent for all $z \in D_R(z_0)$. Furthermore, the Taylor series is unique, with the coefficients given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

Furthermore, the convergence of the series is uniform.

Conversely, any power series of the form

$$\sum_{n=0}^{+\infty} c_n(z-z_0)^n$$

defines an analytic function on the open disk $D_R(z_0)$, where R is the radius of convergence of the power series.

Corollary: If f is analytic in the open connected set Ω and if there exists $\xi \in \Omega$ such that $f^{(n)}(\xi) = 0$ for all $n \in \mathbb{N}$, then $f \equiv 0$ in Ω.

By uniqueness of Taylor series, the corollary is straightforward for the disk $D_R(\xi)$ of convergence of the Taylor series centered in ξ. To complete the proof, we now have to extend the result from $D_R(\xi)$ to Ω. For this purpose, consider the following two sets:

$$E_1 := \{z \in \Omega \mid f^{(n)}(z) = 0 \; \forall n \in \mathbb{N}\} \qquad , \qquad E_2 := \{z \in \Omega \mid \exists n \in \mathbb{N} : f^{(n)}(z) \neq 0\}$$

E_1 and E_2 are such that $E_1 \cap E_2 = \{\emptyset\}$. E_1 is an open set by the existence and uniqueness of Taylor series at any point in Ω. Furthermore, by continuity of f and all its derivatives, E_2 is open as well. Now, since Ω is connected and $\Omega = E_1 \cup E_2$, either $E_1 = \{\emptyset\}$ or $E_2 = \{\emptyset\}$. According to the hypotheses of the theorem, $E_1 \neq \{\emptyset\}$. Therefore $E_2 = \{\emptyset\}$, and $f \equiv 0$, as desired $\square$

Theorem (Differentiation of a power series): Let $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ be an analytic function in the open disk $D_R(z_0)$. The Taylor series for its derivative f' is obtainted by differentiating the power series term by term:

$$f'(z) = \sum_{n=1}^{+\infty} n c_n (z-z_0)^{n-1}$$

Examples

• We will use the uniqueness of a Taylor series to construct the Taylor series of the function

$$f(z) = \frac{1}{(1+z)^2}$$

Let

$$\sum_{n=0}^{+\infty} c_n z^n$$

be that Taylor series. We know that $\forall\, z \in \mathbb{C} \setminus \{-1\}$,

$$f(z)(1+z)^2 = 1 = c_0 + (2c_0 + c_1)z + \sum_{n=2}^{+\infty}(c_{n-2} + 2c_{n-1} + c_n)z^n$$

We thus have $c_0 = 1$, $c_1 = -2$, $c_2 = 3$, and by induction

$$\forall n \in \mathbb{N}\,,\; c_n = (-1)^n(1+n)$$

so that the power series for $1/(1+z)^2$ is

$$\frac{1}{(1+z)^2} = \sum_{n=0}^{+\infty}(-1)^n(1+n)z^n$$

We have

$$\limsup_{n\to\infty} |(-1)^n(1+n)|^{1/n} = 1$$

so the radius of convergence of the infinite series is $R = 1$, as one would expect.

• We use the property of differentiation of Taylor series term by term to calculate the Taylor series for Ln in the disk of radius 1 centered at $z = 1$. Ln is analytic in this disk, and we have $d/dz(\text{Ln}z) = 1/z$. Let us construct the Taylor series for $f(z) = \frac{1}{z}$ centered in $z = 1$:

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{+\infty}(-1)^n(z-1)^n \quad,\ \forall\, z \in \mathbb{C} : |z-1| < 1$$

We conclude that the Taylor series for Ln centered in $z = 1$ is

$$\text{Ln}z = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{(z-1)^n}{n}$$

1.2 Sum and product of two Taylor series

Sum of two Taylor series: Suppose that the two power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad,\quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

have a radius of convergence $R > 0$. Then their sums $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ are analytic functions in the disk $|z-z_0| < R$, and the function $h(z) = f(z) + g(z)$ is analytic in that disk as well, with Taylor series

$$h(z) = \sum_{n=0}^{\infty}(a_n + b_n)(z-z_0)^n$$

Example: $f(z) = e^z$ is an entire function and $g(z) = \text{Ln}(1+z)$ is analytic in the unit disk $D_1(0)$. We have

$$e^z = \sum_{n=0}^{+\infty}\frac{z^n}{n!} \quad,\quad \text{Ln}(1+z) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{z^n}{n}$$

Thus, $h(z) = f(z) - g(z)$ is analytic in the unit disk $D_1(0)$, with Taylor series

$$h(z) = 1 + \sum_{n=1}^{+\infty}\left(\frac{1}{n!} - \frac{(-1)^{n-1}}{n}\right)z^n = 1 + z^2 - \frac{z^3}{6} + \frac{7}{24}z^4 + \dots$$

Product of Taylor series: The product $p(z) = f(z)g(z)$ is also analytic in the disk $|z-z_0| < R$, and therefore has a Taylor series

$$h(z) = \sum_{n=0}^{+\infty} c_n(z-z_0)^n = \sum_{n=0}^{+\infty} a_n(z-z_0)^n \sum_{n=0}^{+\infty} b_n(z-z_0)^n$$

By uniqueness of Taylor series, the coefficients c_n can be expressed in terms of the coefficients a_n and b_n. We have

$$\begin{aligned}
c_0 &= a_0 b_0 \,(= f(z_0)g(z_0)) \\
c_1 &= a_1 b_0 + a_0 b_1 \left(= \frac{f'(z_0)g(z_0) + f(z_0)g'(z_0)}{1!}\right) \\
c_2 &= a_2 b_0 + a_1 b_1 + a_0 b_2 \left(= \frac{f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0)}{2!}\right)
\end{aligned}$$

The general formula is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Example: Let us compute the first few terms of the Taylor series centered in $z = 0$ of $f(z) = e^{-z^2} \sin z$, which is an entire function.

$$e^{-z^2} = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{n!} = 1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6} + \dots \quad , \quad \sin z = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots$$

Hence,

$$e^{-z^2} \sin z = z - \frac{7}{6} z^3 + \frac{27}{40} z^5 + \dots$$

2 Removable singularities

2.1 Riemann's removable singularity theorem

We have said that Cauchy's integral formula applied to functions which were not defined at a finite number of points in Δ, as long as $\lim_{z \to \xi_i} (z - \xi_i) f(z) = 0$ at these points ξ_i. We will now see that Cauchy's integral formula provides a natural way to extend such f to an analytic function on the entire set Δ. In other words, the ξ_i are *removable* singularities.

Theorem: Suppose that f is analytic in the open connected set Ω' obtained by omitting the point ξ from an open connected set Ω. There exists an analytic function in Ω which coincides with f in Ω' iff $\lim_{z \to \xi} (z - \xi) f(z) = 0$. The extended function is uniquely determined.

Proof: If the extended function exists, it is continuous in ξ, which guarantees uniqueness.
Likewise, by continuity of the extended function $\tilde{f}$, $\lim_{z \to \xi} (z - \xi) f(z) = \lim_{z \to \xi} (z - \xi) \tilde{f}(z) = 0$, which takes care of the necessary condition in the theorem.

For the sufficient condition, consider a circle centered at ξ and such that the circle C and the disk Δ corresponding to its interior are contained in Ω. For $z \neq \xi$ in Δ, we construct

$$F(\zeta) := \frac{f(\zeta) - f(z)}{\zeta - z}$$

F has two singularities in Δ: $\zeta = z$ and $\zeta = \xi$. We have

$$\lim_{\zeta \to z} (\zeta - z) F(\zeta) = 0$$

by continuity of f in z. We also have

$$\lim_{\zeta \to \xi} (\zeta - \xi) F(\zeta) = 0$$

by the hypothesis of the theorem. Therefore, applying Cauchy's theorem to F,

$$\int_C F(\zeta) d\zeta = 0$$

Hence, for any $z \neq \xi$ in Δ,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1}$$

Now, we know from the lemma in Section 1.3 of Lecture 5 that the right-hand side of (1) is an analytic function of z throughout the inside of C. It is therefore continuous in ξ, with value

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \xi} d\zeta$$

In other words,

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \ , \ \ \forall z \in \Omega \tag{2}$$

is the desired analytic extension of f in the whole open connected set Ω.

2.2 Taylor's theorem

Let us apply the previous result to the function

$$F(z) := \frac{f(z) - f(\xi)}{z - \xi}$$

where $(z, \xi) \in \Omega^2$, with $z \neq \xi$, Ω is an open connected set, as before, and f is analytic on Ω.
Observe that

$$\lim_{z \to \xi} (z - \xi) F(z) = 0 \qquad , \qquad \lim_{z \to \xi} F(z) = f'(\xi)$$

By the previous theorem, there exists an analytic function f_1 on Ω such that

$$\begin{cases} f_1(z) = F(z) & \text{if } z \neq \xi \\ f_1(\xi) = f'(\xi) \end{cases}$$

$\forall\, z \in \Omega$, we may thus write

$$f(z) = f(\xi) + (z - \xi) f_1(z)$$

This expansion for f can also be applied to f_1: there exists an analytic function f_2 on Ω such that

$$f_1(z) = f_1(\xi) + (z - \xi) f_2(z)$$

with

$$\begin{cases} f_2(z) = \frac{f_1(z) - f_1(\xi)}{z - \xi} & \text{if } z \neq \xi \\ f_2(\xi) = f_1'(\xi) \end{cases}$$

Continuing the recursion, we can write the general form

$$f_{n-1}(z) = f_{n-1}(\xi) + (z - \xi) f_n(z)$$

In this process, we obtained the following expansion for f:

$$f(z) = f(\xi) + (z - \xi) f_1(\xi) + (z - \xi)^2 f_2(\xi) + \ldots + (z - \xi)^{n-1} f_{n-1}(\xi) + (z - \xi)^n f_n(z)$$

Furthermore, by direct differentiation at $z = \xi$, we have

$$f^{(n)}(\xi) = n! f_n(\xi)$$

We have just prove Taylor's theorem, stated below:

Theorem: If f is analytic in an open connected set Ω containing ξ, it is possible to write

$$f(z) = f(\xi) + f'(\xi)(z - \xi) + \frac{f''(\xi)}{2}(z - \xi)^2 + \ldots + \frac{f^{(n-1)}(\xi)}{(n-1)!}(z - \xi)^{n-1} + (z - \xi)^n f_n(z) \tag{3}$$

where f_n is analytic in Ω.

Note that Taylor's formula, given by Eq.(3), is *not* a Taylor series. It is very useful nonetheless, especially because there is a simple expression for f_n in terms of f:

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\xi)^n(\zeta-z)}d\zeta \tag{4}$$

To see why (4) holds, we start with Cauchy's integral formula,

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{(\zeta-z)}d\zeta$$

and we represent f_n in the integrand using Taylor's formula. When we do so, the first term is

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\xi)^n(\zeta-z)}d\zeta$$

The other terms all have the following form, to within a constant factor:

$$g_k(\xi) = \int_C \frac{d\zeta}{(\zeta-\xi)^k(\zeta-z)} \quad 1 \le k \le n-1$$

Observe first that one may write

$$g_k(\xi) = \int_C \frac{\varphi(\zeta)}{(\zeta-\xi)^k}d\zeta \quad 1 \le k \le n-1$$

with φ continuous on C. Hence, from the lemma in Section 1.3 of Lecture 5, $\forall\, k \in [\![2, n-1]\!]$, $g_k'(\xi) = k g_{k+1}(\xi)$. So all we need to do is evaluate

$$g_1(\xi) = \int_C \frac{d\zeta}{(\zeta-\xi)(\zeta-z)} = \frac{1}{z-\xi}\int_C \left(\frac{1}{\zeta-z} - \frac{1}{\zeta-\xi}\right)d\zeta = \frac{1}{z-\xi}(2\pi i - 2\pi i) = 0$$

where we recognized the definition of the winding number to derive the third equality above. We conclude that $g_1(\xi) = 0$, and thus $g_k(\xi) = 0$ for $k \in [\![2, n-1]\!]$, from which the formula (4) for f_n follows.

3 Zeros and poles

3.1 Zeros of a function

Let f be an analytic function in Ω which is not identically zero, and $\xi \in \Omega$. From what we have just seen, there exists a first integer N such that $f^{(N)}(\xi) \neq 0$. Then, by Taylor's formula, we can write

$$f(z) = (z-\xi)^N f_N(z)$$

with f_N analytic and such that $f_N(\xi) \neq 0$. We say that ξ is a *zero of order* N of f.

Observe that f_N is continuous, so $\exists\, \delta > 0$ such that $\forall\, z$ such that $0 < |z-\xi| < \delta$, $f(z) \neq 0$: the zeros of f are isolated. This can be reformulated with the following theorem:

Identity Theorem: If f and g are analytic in Ω, and if $f = g$ on a set which has an accumulation point in Ω, then $\forall\, z \in \Omega$, $f(z) = g(z)$.
The theorem is immediate by looking at the Taylor formula for $f - g$, as long as we remember what an accumulation point is:

- A point z of a subset S is called an isolated point of S if there exists a neighborhood of z whose intersection with S reduces to the point z
- An accumulation point is a point of $\overline{S}$ which is not an isolated point.

A trivial yet important consequence of the identity theorem is as follows:
If f is analytic in Ω and identically zero in a nonempty connected open subset of Ω, then $f \equiv 0$ in Ω.
Likewise, if f is identically zero on an arc in Ω which does not reduce to a point, $f \equiv 0$ in Ω.

3.2 Poles of a function

Consider a function f which is analytic in a neighborhood of ξ, but perhaps not in ξ itself. ξ is then called an *isolated singularity*.

If $\lim_{z\to\xi} f(z) = \infty$, ξ is said to be a *pole of* f.
By continuity, there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(\xi)$ with $z \neq \xi$. Thus, $g(z) := 1/f(z)$ is analytic for all z such that $0 < |z - \xi| < \delta$. Furthermore, g can be analytically extended on $D_\delta(\xi)$, with $g(\xi) = 0$ since $\lim_{z\to\xi} (z - \xi)g(z) = 0$.
The order of the pole of f in ξ is the order N of the zero of g in ξ. We can write

$$f(z) = \frac{f_N(z)}{(z-\xi)^N} \quad , \quad 0 < |z - \xi| < \delta$$

with f_N analytic and nonzero in a neighborhood of ξ.

Definition: A function which is analytic in an open connected set Ω except for isolated poles is called a *meromorphic function*.

If f has a pole of order N at ξ, then we can use Taylor's formula to write:

$$(z-\xi)^N f(z) = a_N + a_{N-1}(z-\xi) + \ldots + a_1(z-\xi)^{N-1} + \varphi(z)(z-\xi)^N$$

with φ analytic at $z = \xi$. Hence, for $z \neq \xi$, we may write

$$f(z) = \frac{a_N}{(z-\xi)^N} + \frac{a_{N-1}}{(z-\xi)^{N-1}} + \ldots + \frac{a_1}{z-\xi} + \varphi(z)$$

where the sum of the terms in blue is called *the singular part of* f *at* ξ.

3.3 Essential singularity

Let f be analytic in a disk $0 < |z - \xi| < \delta$ with the center ξ removed.

(i) If $\lim_{z\to\xi} f(z)$ exists or if $\lim_{z\to\xi} (z-\xi)f(z) = 0$, then ξ is a removable singularity, and f extends to an analytic function on the whole disk $|z - \xi| < \delta$

(ii) If $\lim_{z\to\xi} f(z) = \infty$, ξ is said to be a pole. In this case, $f(z) = (z - \xi)^{-N} f_N(z)$ with $N \in \mathbb{N}^*$ the order of the pole, f_N analytic in a neighborhood of ξ, and $f_N(\xi) \neq 0$.

(iii) If neither (i) nor (ii) holds, ξ is said to be an *essential singularity*.

Example: $f(z) = \exp(1/z)$ has an essential singularity at $\xi = 0$.

The behavior of a function near an essential singularity is quite extreme, as illustrated by the following theorem.

Casorati-Weierstrass theorem: An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.

Proof: Suppose the statement is false: $\exists\, z_0 \in \mathbb{C}$ and $\delta > 0$ and $\epsilon > 0$ such that

$$|f(z) - z_0| > \epsilon \quad \text{for all } z \text{ such that} \quad |z - \xi| < \delta$$

Thus,

$$\lim_{z\to\xi} \frac{f(z) - z_0}{z - \xi} = \infty$$

so that the function

$$g(z) := \frac{f(z) - z_0}{z - \xi}$$

has a pole at $z = \xi$. We may then write $g(z) = (z-\xi)^{-N} g_N(z)$ with $N \in \mathbb{N}^*$ and g_n analytic in a neighborhood of ξ. In other words,

$$f(z) = (z-\xi)^{1-N} g_N(z) + z_0$$

If $N = 1$, f has a removable singularity at $z = \xi$.
If $N > 1$, $f - z_0$ has a pole at $z = \xi$, and so does f.
Both possibilities are excluded by the hypothesis of the theorem, so the statement must be true. □