1 Laurent series

1.1 Analytic functions on an annulus

Let $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ be an annulus. For each analytic function $f : A \to \mathbb{C}$ there are analytic functions $F_1 : \{z \in \mathbb{C} : |z| > R_1\} \to \mathbb{C}$ and $F_2 : \{z \in \mathbb{C} : |z| < R_2\} \to \mathbb{C}$ such that $\forall z \in A$, $f(z) = F_2(z) - F_1(z)$.

Proof: Let $z_0 \in A$, and define for all $z \in A \setminus \{z_0\}$

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

g can be extended to an analytic function on all of A. Let R such that $R_1 < R < R_2$, and the circle $C_R(0)$ with radius R centered in 0. The quantity

$$U = \frac{1}{2\pi i} \int_{C_R(0)} g(z)dz$$

is independent of R, by an application of Cauchy's theorem.

Indeed, consider a second circle $C_{R'}(0)$ centered in 0 and contained in A and the contour γ made of the piecewise differentiable green, red and black arcs shown in Figure 1. By Cauchy's theorem,

$$\int_\gamma g(z)dz = 0 \quad \Leftrightarrow \quad \int_{\gamma_1} g(z)dz = \int_{\gamma_2} g(z)dz + I_\epsilon$$

where I_ϵ is the contribution from the two black horizontal segments separated by a distance ϵ.

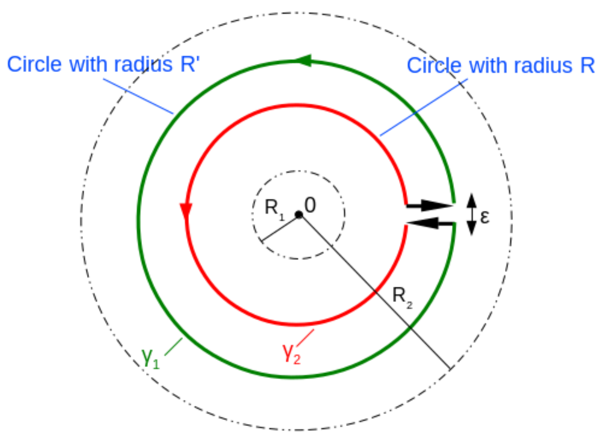

Figure 1: Arcs used to prove that the value of U is independent of the radius of the circle C

Since g is continuous in A, $\lim_{\epsilon \to 0} I_\epsilon = 0$, which proves that

$$\int_{C_{R'}(0)} g(z)dz = \int_{C_R(0)} g(z)dz$$

Now, by direct evaluation, we may write

$$U = \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(z)}{z - z_0} dz - f(z_0) n(C_R(0), z_0)$$

We may thus write

$$\begin{cases} U = \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(z)}{z-z_0} dz \quad , \quad R_1 < R < |z_0| \\ U = \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(z)}{z-z_0} dz - f(z_0) \quad , \quad |z_0| < R < R_2 \end{cases} \tag{1}$$

We define, for R such that $R_1 < R < |z_0|$

$$F_1(z_0) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(z)}{z - z_0} dz$$

F_1 is analytic on $\{z_0 : R < |z_0|\}$, and its value is independent of $R \in (R_1, |z_0|)$, so it is an analytic function on $\{z_0 : R_1 < |z_0|\}$.
Likewise, for R such that $|z_0| < R < R_2$, we define

$$F_2(z_0) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(z)}{z - z_0} dz$$

which is analytic on $\{z_0 : |z_0| < R_2\}$, following an argument which parallels the argument given just above.
Finally, from (1), we find that $f(z_0) = F_2(z_0) - F_1(z_0)$ $\square$

Observe that the proof and result given above can be generalized to an annulus centered in $a \in \mathbb{C}$ instead of just 0 without difficulty. We will use this fact in the next section.

1.2 Laurent series

Any analytic function f on $A = \{z \in \mathbb{C} : R_1 < |z - a| < R_2\}$ can be developed in a power series of the form

$$f(z) = \sum_{n=-\infty}^{n=+\infty} c_n (z - a)^n \tag{2}$$

The series above, called a Laurent series, converges *uniformly* on A. Moreover,

$$\forall n \in \mathbb{Z} \ , \ n(\gamma, a) c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz$$

Proof: Without loss of generality, we translate A so that $a = 0$. From our previous result, we know that $\forall z \in A$, $f(z) = F_2(z) - F_1(z)$, with F_1 and F_2 as given above.
F_2 is analytic on the disk $|z| < R_2$, so it has a power series

$$F_2(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges uniformly on this disk.
For F_1, we consider the function $G(z) := F_1(1/z)$. From the integral representation of F_1, it is clear that $F_1(z) \to 0$ as $z \to \infty$. Therefore, $G(z) \to 0$ as $z \to 0$, so G can be viewed as an analytic function on the disk $|z| < 1/R_1$. On that disk, one can write $G(z) = \sum_{n=1}^{\infty} b_n z^n$, and the series converges uniformly on the disk.
In other words, the series

$$F_1(z) = \sum_{n=1}^{\infty} b_n z^{-n}$$

converges uniformly on $\{z : |z| > R_1\}$.
We conclude that $\forall z \in A$, $f_z = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n z^{-n}$, with each sum converging uniformly on A.

Using the uniform convergence, we may then write:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz = \sum_{k=-\infty}^{+\infty} \frac{c_k}{2\pi i}\int_\gamma (z - a)^{k-n-1} dz$$

The integrals on the right-hand side are zero, except when $k - n - 1 = -1$. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz = c_n n(\gamma, a) \qquad \square$$

1.3 Laurent series for isolated singularities

Let f be holomorphic on $\Omega \setminus \{a\}$, where Ω is an open connected set, and a an isolated singularity. Its Laurent series on the punctured disk $0 < |z-a| < R$ is given by $f(z) = \sum_{n=-\infty}^{n=+\infty} c_n (z-a)^n$.

- f has a removable singularity at a iff $c_n = 0$ for all n such that $n < 0$.
- f has a pole of order N at a iff $c_n = 0$ for all $n < -N$ and $c_{-N} \neq 0$.
- f has an essential singularity at a iff $c_n \neq 0$ for infinitely many negative values of n.

The proof of this last point is left for the reader.

2 Examples

2.1 Example 1

Let us consider the function

$$f(z) = e^{\frac{1}{z}}$$

For any $R \in \mathbb{R}^+$, the function $g(z) = \frac{1}{z}$ is analytic on the punctured disk $\{z \in \mathbb{C} : 0 < |z| < R$, and so is $f(z) = e^{g(z)}$. Thus, f has a Laurent series in this punctured disk. We know that $h(z) = e^z$ is an entire function with Taylor series

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

For $z \neq 0$, let $w = \frac{1}{z} \Leftrightarrow z = \frac{1}{w}$.

$$e^w = \sum_{n=0}^{+\infty} \frac{w^n}{n!}$$

which means

$$e^{\frac{1}{z}} = \sum_{n=0}^{+\infty} \frac{1}{n! z^n}$$

This is the desired Laurent series. We observe that $c_n \neq 0$ for infinitely many negative values of n. This confirms that f has an essential singularity at $z = 0$, as we had seen in the previous lecture.

2.2 Example 2

Let us consider the function

$$f(z) = \frac{1}{(z-2)(z-1)}$$

which has two simples poles: $z = 1$ and $z = 2$.

We may rewrite f as

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

• For $|z| < 1$,

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$$

$$\frac{1}{z-2} = -\frac{1}{2-z} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{+\infty} \left(\frac{z}{2}\right)^n$$

Hence, for $|z| < 1$, f has the power series

$$f(z) = -\frac{1}{2}\sum_{n=0}^{+\infty} \left(\frac{z}{2}\right)^n + \sum_{n=0}^{+\infty} z^n = \sum_{n=0}^{+\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n$$

- For $1 < |z| < 2$,

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{+\infty}\frac{1}{z^n}$$

Therefore, f has the following Laurent series on the annulus $1 < |z| < 2$:

$$f(z) = -\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n - \sum_{n=1}^{+\infty}\frac{1}{z^n}$$

- For $|z| > 2$, we can write

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{2}{z}\right)^n$$

so that f has the following Laurent series on the annulus $|z| > 2$:

$$\frac{1}{z}\sum_{n=0}^{+\infty}\frac{2^n-1}{z^n} = \sum_{n=1}^{+\infty}\frac{2^{n-1}-1}{z^n}$$