1 The general form of Cauchy's theorem

1.1 Chains

Let Ω be an open set in $\mathbb{C}$. A chain in Ω is a finite collection $\gamma_j \; : [a_j, b_j] \; \rightarrow \Omega \; , \; j = 1, \ldots, N$ of piecewise continuously differentiable curves in Ω.
Writing $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ for a given chain, we can integrate a continuous function f in Ω along Γ as follows:

$$\int_\Gamma f(z)dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)dz$$

1.2 Cycles

A cycle in Ω is a chain $\Gamma = \sum_{j=1}^{N} \gamma_j$ where each point $z \in \mathbb{C}$ is an initial point of just as many of the γ_j as it is a terminal point. In other words, a cycle is a finite sum of closed curves.

As an illustration, the index of a point z with respect to the cycle Γ is

$$n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z} = \sum_{j=1}^{N} \int_{\gamma_j} \frac{d\zeta}{\zeta - z}$$

Observe that the integrals in the sum on the right-hand side of the last equality above may not be over closed curves.

1.3 Simple connectivity and homology

1.3.1 Simply connected sets in $\mathbb{C}$

We start this section with an unusual definition for simple connectedness. Its weakness is that it is not general, in the sense that it cannot be used in $\mathbb{R}^n$ with $n \geq 3$. However it can be shown that for $\mathbb{C}$, it is equivalent to the more common definition, which says that any simple closed curve can be shrunk to a point continuously in the set. And the advantage of our unusual definition is that it is more convenient for the statement of the general form of the Cauchy-Goursat theorem.

Definition: An open connected set $\Omega \subset \mathbb{C}$ is said to be simply connected if its complement with respect to $\hat{\mathbb{C}}$ is connected.

Note: In this definition, it is important to stress that the complement is with respect to $\hat{\mathbb{C}}$ and not just $\mathbb{C}$. Missing this point could lead you to easily find counterexamples which do not agree with the definition.

Theorem: An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$

Proof: • Let us start with the necessary condition: for any cycle $\gamma \in \Omega$, the complement of Ω in $\hat{\mathbb{C}}$ must be in one of the regions determined by γ (interior or exterior), since this complement is connected. Since $\{\infty\}$ belongs to this complement, this must be the unbounded region defined by γ. From Lecture 5, we thus know that $n(\gamma, z) = 0 \; \forall z \notin \Omega$.

• We prove the sufficient condition by direct construction. Specifically, we will show that if a region Ω is not simply connected, then one can construct a cycle γ in Ω and find a point z_0 which does not belong to Ω such that $n(\gamma, z_0) \neq 0$.

Let us assume that the complement of Ω in $\hat{\mathbb{C}}$ is $A \cup B$, with A and B disjoint closed sets, with a shortest distance $\delta > 0$ between the two sets. Let us say that B is the unbounded set, so A is bounded. We cover A with a net of squares S whose sides have length $l < \delta/\sqrt{2}$, constructed in such a way that $z_0 \in A$ lies at the center of a square, as shown in Figure 1.

Consider the cycle $\gamma = \sum_j \partial S_j$, where ∂S_j is the boundary curve of each square S_j, and where the sum is taken over the net covering A.

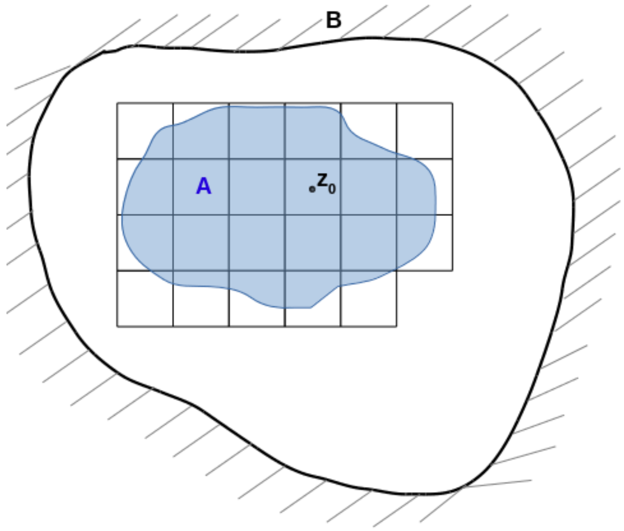

Figure 1: Net of squares covering the set A without intersecting the set B

Observe first that $n(\gamma, z_0) = 1$ since z_0 belongs to only one of the squares in the net.
Furthermore, it is clear that γ does not belong to B. Now, the key is to realize that γ does not belong to A either, in the sense that there exists a cycle $\tilde{\gamma}$ contained in Ω such that $n(\tilde{\gamma}, z_0) = n(\gamma, z_0) = 1$. Indeed, $\tilde{\gamma}$ is directly obtained from γ by observing that in the integral corresponding to $n(\gamma, z_0)$, all the sides of the squares contained in A are traversed exactly twice, in opposite directions, and therefore cancel. This concludes our proof $\square$

1.3.2 Homology

Definition: A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, z) = 0$ for all z in the complement of Ω in $\hat{\mathbb{C}}$.
One writes $\gamma \sim 0 \pmod{\Omega}$, or often $\gamma \sim 0$ when it is clear that one is talking about Ω.
$\gamma_1 \sim \gamma_2$ means $\gamma_1 - \gamma_2 \sim 0$

Note that with this notation, the previous theorem can be written as
Theorem: An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $\gamma \sim 0$ for all γ in Ω.

1.4 The general form of the Cauchy-Goursat theorem

We now have all the tools required to give the Cauchy-Goursat theorem in its most general form.

Theorem (General form of the Cauchy-Goursat theorem): If f is analytic in the open set Ω, then $\int_\gamma f(z)dz = 0$ for every cycle γ which is homologous to zero in Ω.

For the sake of time, we will not provide a proof of this theorem in these notes. The interested reader can find an elegant proof, first proposed by John Dixon in the *Proceedings of the American Mathematical Society*, Volume 29, Number 3, August 1971 in Lecture 9 of my notes for the PhD level class, which can be found here: `https://www.math.nyu.edu/~cerfon/complex_notes/Lecture_9.pdf`

Corollary 1: If f is analytic in a simply connected open set Ω, then $\int_\gamma f(z)dz = 0$ for all cycles in Ω.

This follows directly from the Cauchy-Goursat theorem, and the theorem in page 1 of these notes. This corollary corresponds to the "simple form" of the Cauchy-Goursat theorem we presented without proof in Section 4.4 of Lecture 4.

Corollary 2: If f is analytic and nonzero in a simply connected open region Ω, then it is possible to define single-valued analytic branches of $\ln[f(z)]$ and $\sqrt[n]{f(z)}$ in Ω.

Indeed, by the Cauchy-Goursat theorem we know that

$$\int_\gamma \frac{f'(z)}{f(z)}dz = 0$$

for all cycles in Ω. We then know that there exists an analytic function F such that $F'(z) = f'(z)/f(z)$ $\forall z \in \Omega$.
In other words,

$$\frac{d}{dz}\left[f(z)e^{-F(z)}\right] = 0 \ \Leftrightarrow\ f(z) = Ae^{F(z)}\ ,\ A \in \mathbb{C}^*$$

Now, choose $z_0 \in \Omega$ and one of the infinitely many values of $\ln[f(z_0)]$.

$$\exp\left[F(z) - F(z_0) + \ln[f(z_0)]\right] = \frac{f(z)}{A}e^{-F(z_0)}f(z_0) = f(z)$$

We can therefore define a single-valued, analytic branch of the logarithm of f as

$$\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$$

The definition of $\sqrt[n]{f}$ follows from this result, as $\forall z \in \Omega$ we write $\sqrt[n]{f} = \exp\left[\frac{1}{n}\ln(f(z))\right]$

2 The residue theorem

2.1 Residue of a function at a point

Definition: Consider a function f which is analytic in an open connected set Ω except for the isolated singularity at a. Consider a circle C centered in a and contained in Ω. Let

$$P = \int_C f(z)dz$$

If we set $R = \frac{P}{2\pi i}$, the function

$$g(z) := f(z) - \frac{R}{z-a}\quad ,\quad \forall z \in \Omega \setminus \{a\}$$

is such that

$$\int_C g(z)dz = 0$$

R as defined above is called the *residue of f at a*:

$$\text{Res}_{z=a}f(z) = \frac{1}{2\pi i}\int_{C_R(a)} f(z)dz$$

Of course, the definition only makes sense if it is independent of the choice of the radius $R > 0$ of the circle C. By the Cauchy-Goursat theorem, following exactly the same procedure as we have done in the first page of Lecture 7, this is not too hard to see.

2.2 The residue theorem

Consider a function f which is analytic in the open connected set Ω except for finitely many singularities a_j. Let γ be a cycle in $\Omega' = \Omega \setminus \{a_j\}_{j=1,\ldots,N}$ which is homologous to zero with respect to Ω. Then $\gamma \sim \sum_{j=1}^N n(\gamma, a_j)C_j \pmod{\Omega'}$, where C_j is any circle centered in a_j and contained in Ω'.
By the general form of the Cauchy-Goursat theorem, we can thus write

$$\int_\gamma f(z)dz = \sum_{j=1}^N n(\gamma, a_j)\int_{C_j} f(z)dz$$

$$\Leftrightarrow \frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{j=1}^N n(\gamma, a_j)\text{Res}_{z=a_j}f(z) \tag{1}$$

Theorem (Residue theorem): Let f be analytic except for isolated singularities a_j in an open connected set Ω. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j)\text{Res}_{z=a_j} f(z) \tag{2}$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j. The sum (2) is finite.

2.3 Computing residues

As one may expect, the residue theorem is particularly convenient to use when γ is such that $\forall a_j$, $n(\gamma, a_j) =$ 0 or 1.

More importantly, it is only useful as a tool for integration if there is a simple method to compute residues. Returning to Lecture 7 and Laurent series, we see that the definition of the residue of f at a coincides with the coefficient c_{-1} of the Laurent series of f centered in a, i.e. the coefficient of $1/(z-a)$ in that Laurent series.

For the situation in which f has a pole of order N at a, we can derive another method for calculating the residue of f at a, which is often more convenient and faster.
$g(z) = (z-a)^N f(z)$ is analytic in a neighborhood of a. Integrating along a circle C centered in a and in that neighborhood, we may write

$$g^{(N-1)}(a) = \frac{(N-1)!}{2\pi i}\int_C \frac{g(z)}{(z-a)^N}dz = (N-1)!\text{Res}_{z=a} f(z)$$

Hence,

$$\text{Res}_{z=a} f(z) = \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\left[(z-a)^N f(z)\right]\Bigg|_{z=a} \tag{3}$$

In particular, if $f(z) = g(z)/h(z)$ and h has a simple zero at a and $g(a) \neq 0$,

$$\text{Res}_{z=a} f(z) = \frac{g(a)}{h'(a)}$$

Example: Use the residue theorem to compute

$$\oint_{|z|=1} \frac{e^{iz}}{z^2}dz$$

where the circle is traversed in the counterclockwise direction.

3 Additional local properties of analytic functions

In previous lectures, we saw some fundamental local properties of analytic functions, such as the fact that any analytic function can be locally expanded as a power series with a finite radius of convergence, or the fact that the modulus of a nonconstant analytic function cannot have a maximum inside an open connected set on which the function is defined and analytic.

Now that we introduced the residue theorem, we will be able to efficiently derive a new set of important local properties.

3.1 The argument principle

Theorem (The argument principle): If f is meromorphic in an open connected set Ω, with zeros a_j and poles b_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \tag{4}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros and poles. The sums in (4) are finite, and multiple zeros and poles have to be repeated as many times as their order indicates.

Proof: Let us first assume that the function has a finite number of zeros and poles, and call K that number. Consider the orders N_j of the zeros and poles z_j of f in Ω. $N_j > 0$ if z_j is a zero of f, $N_j < 0$ if z_j is a pole of f. Let

$$g(z) := f(z)\Pi_{j=1}^{K}(z - z_j)^{-N_j}$$

g only has removable singularities in Ω, so we can view it as analytic in Ω. Furthermore, g does not have zeros inside Ω. Writing $f(z) = g(z)\Pi_{i=j}^{K}(z - z_j)^{N_j}$ and taking the logarithmic derivative of that equality for $z \neq z_j$, we find

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{K} \frac{N_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

We integrate this equality along any cycle γ homologous to zero in Ω and which does not pass through any z_j. By the Cauchy-Goursat theorem, $\int_\gamma g'(z)/g(z) = 0$, and by the definition of the index of a point with respect to a curve applied to the remainder of the right-hand side (or by the residue theorem if you prefer to see it this way), we get the desired result.

The proof can be extended to the situation in which the function f may have an infinite number of zeros and/or poles. Consider for example the situation in which f has infinitely many zeros in Ω. Since γ is inside Ω, it is contained in a closed set Ω' inside Ω. Now, since f is not identically zero, it can only have finitely many zeros inside Ω'. This result follows from a combination of the Bolzano-Weierstrass theorem and the identity theorem. Therefore, the formula (4) holds inside Ω'. It then holds in Ω too, since for the zeros ζ_j of f outside of Ω', $n(\gamma, \zeta_j) = 0$. A similar argument can be easily constructed if f has an infinite number of poles. Thus Eq.(4) remains true in these cases as well, with the sums still finite $\Box$

Intuitive interpretation of the argument principle

Observe that the integral on the left of (4) can be represented as

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}dt = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}dt = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w}$$

$f \circ \gamma$ is a closed curved Γ. The equality (4) in the theorem can thus be interpreted as the equality

$$n(\Gamma, 0) = \sum_j n(\gamma, \zeta_j) - \sum_k n(\gamma, b_k)$$

where $n(\Gamma, 0)$ represents the number of times the closed curve Γ, which is the image of γ by f, goes around the origin.

A very common case is the one for which γ is a simple closed curve C oriented counterclockwise. In that case, the argument principle takes the simple form:

$$n(\Gamma, 0) = N - P$$

where N and P denote the number of zeros and poles of f inside C, with each zero and pole counted as many times as its order indicates.

This formula is at the heart of a number of numerical methods to locate the zeros and poles of an analytic function. This is because even fairly inaccurate algorithms for the calculation of $n(\Gamma, 0)$ will give results close enough to an integer value that one can directly infer the corresponding exact integer value.

The name "argument principle" can be given the following intuitive – although not at all rigorous – interpretation:

$$\text{“}\ \frac{dw}{w} = d(\ln w) = d(\ln|w| + i\arg w)\ \text{”}$$

Note the quotes around these equalities, which should be seen as formal equalities and nothing else. For any curve that does not pass through 0, $\ln|w|$ is well defined, so by the fundamental theorem of calculus the

contribution of the real part in the formal equalities above to the integral is zero when one integrates over a closed curve.
Let me stress, once more, that this is just intended to provide an intuition, but it is not fully rigorous.

Example: Let $f(z) = z^2 + z$, $\Gamma_1 = f \circ \gamma_1$ with γ_1 the circle of radius 1/2 centered in 0, and $\Gamma_2 = f \circ \gamma_2$ with γ_2 the circle of radius 2 centered in 0.
We have $n(\Gamma_1, 0) = 1 - 0 = 1$, and $n(\Gamma_2, 0) = 2 - 0 = 0$.
Note that for this example, we could not apply the argument principle for the unit circle centered in 0 since $f(-1) = 0$.

3.2 Rouché's theorem

Rouché's theorem can be viewed as a corollary of the argument principle. It can be stated as follows.

Theorem (Rouché's theorem): Let γ be a cycle which is homologous to zero in the open connected set Ω and such that $n(\gamma, z)$ is either 0 or 1 for all $z \in \Omega$ such that $z \notin \gamma$. Suppose that f and g are analytic in Ω, and that $\forall z \in \gamma$, $|f(z) - g(z)| < |f(z)|$. Then f and g have the same number of zeros *enclosed* by γ.

Proof: From the hypotheses of the theorem, we know that $\forall\, z \in \gamma$, $f(z) \neq 0$ and $g(z) \neq 0$. Along γ, we can therefore consider the function $\psi(z) := g(z)/f(z)$. ψ is such that

$$\forall\, z \in \gamma,\ |\psi(z) - 1| < 1$$

Hence,

$$\int_\gamma \frac{\psi'(z)}{\psi(z)} = \int_\Gamma \frac{d\zeta}{\zeta} = 2\pi i n(\Gamma, 0) = 0$$

where we have used the change of variable $\zeta = \psi(z)$, $\Gamma = \psi(\gamma)$ to derive the first equality.
Now, let N_g be the number of zeros of g inside γ, and N_f the number of zeros of f inside γ. By the argument principle,

$$0 = \int_\gamma \frac{\psi'(z)}{\psi(z)} dz = N_g - N_f \quad \Leftrightarrow \quad N_f = N_g \quad \square$$

Typical example of the use of Rouché's theorem

Consider the polynomial $z^4 - 6z + 3$. How many zeros does it have in the annulus between $|z| = 1$ and $|z| = 2$?

Start with $\gamma_1 : \ |z| = 2$, and take $f_1(z) = z^4$, $g_1(z) = z^4 - 6z + 3$.

$$\forall z \in \gamma_1 \ ,\ |f_1(z) - g_1(z)| = |6z - 3| \leq 15 < 16 = |f_1(z)|$$

Hence both f_1 and g_1 have 4 zeros inside $|z| = 2$.

Now consider $\gamma_2 : \ |z| = 1$, and define $f_2(z) = -6z$, $g_2(z) = z^4 - 6z + 3$

$$\forall z \in \gamma_2 \ ,\ |f_2(z) - g_2(z)| = |z^4 + 3| \leq 4 < 6 = |f_2(z)|$$

So both f_2 and g_2 have 1 zero inside $|z| = 1$
We conclude that $z^4 - 6z + 3 = 0$ has 3 roots in the annulus.

3.3 Open mapping theorem

Another important local property of analytic functions, which can be viewed as a corollary of Rouché's theorem, can be expressed as follows.

Theorem (Open mapping theorem): A nonconstant analytic function maps open sets to open sets.

Proof: Let us consider a nonconstant analytic function f on an open set Ω, $z_0 \in \Omega$, and $w_0 = f(z_0)$. We need to show that there is a neighborhood $D_\delta(w_0)$ such that $D_\delta(w_0) \subset f(\Omega)$, i.e. that

$$\forall\, w \text{ s.t. } |w - w_0| < \delta \ , \ \exists\, z \in \Omega \text{ s.t. } w = f(z)$$

.

The idea of the proof is to look for zeros of $f(z) - w$ for w close to w_0, knowing that $f(z) - w_0$ has the zero z_0.

Step 1: Let us take ϵ small enough that that $\overline{D_\epsilon}(z_0) \subset \Omega$ and that the closed disk does not contain other zeroes of $F(z) := f(z) - w_0$. It is possible to find such a disk, since the zeros of analytic functions are isolated. In particular, we can write

$$\forall\, z \in \ C_\epsilon(z_0) \ , \ F(z) \neq 0$$

Step 2: The function $|F(z)|$ is continuous on $C_\epsilon(z_0)$ and therefore achieves its minimum on $C_\epsilon(z_0)$ since $C_\epsilon(z_0)$ is compact. Let

$$\delta = \min\{|F(z)| \ , \ z \in C_\epsilon(z_0)\}$$

By construction of $C_\epsilon(z_0)$, $\delta > 0$.
Step 3: For all $w \in \mathbb{C}$ such that $|w - w_0| < \delta$, we define

$$g_w(z) = f(z) - w$$

We have, for all $z \in C_\epsilon(z_0)$

$$|F(z) - g_w(z)| = |f(z) - w_0 - f(z) + w| = |w - w_0| < |F(z)|$$

Hence, for any $w \in \mathbb{C}$ such that $|w - w_0| < \delta$, g_w has the same number of zeros as F inside $D_\epsilon(z_0)$ by Rouché's theorem, and therefore at least one. This concludes our proof □