One of the important applications of the calculus of residues is the evaluation of definite and improper integrals along the *real axis* one encounters in analysis and in applied mathematics. The idea is to extend the integration path into the complex plane and use the residue theorem to evaluate the resulting integral. This will often significantly simplify the calculation of the desired integral. We show this through a few examples.

1 Example 1

Let us evaluate the improper integral

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2}dx \ , \ \ t \in \mathbb{R}$$

The first question one may ask is the following: does the expression above make any sense? In other words, does the integral converge? The answer is yes, by absolute convergence.

To evaluate this integral in a slick way, let us use complex analysis and the residue theorem. We define

$$f(z) := \frac{e^{itz}}{1+z^2} \ , \ \ t \in \mathbb{R}$$

f has simple poles at $z = \pm i$.
The idea is to close the contour along the real line in such a way as to benefit from the exponential decay of e^{itz} and use the residue theorem for the value of the integral over the closed contour.

• If $t \geq 0$, then we consider a family of contours γ_R from $-R$ to R along the real axis, closed by the semi-circle centered in 0 and with radius R in the upper half-plane, as shown in Figure 1.

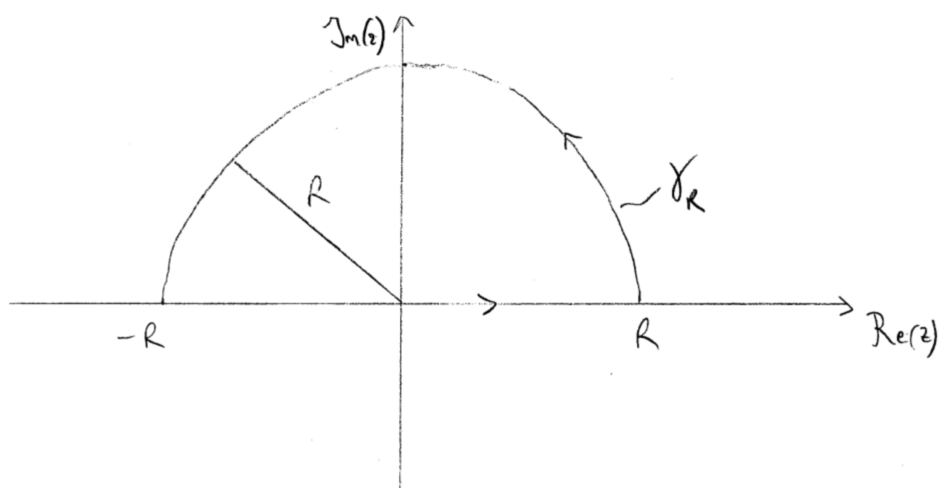

Figure 1: Contour in the complex plane for the first case in Example 1

According to the residue theorem, we can write

$$\int_{\gamma_R} f(z)dz = 2\pi i \text{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}$$

Now, we can also write

$$\int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + iR\int_0^{\pi} \frac{e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}e^{i\theta}d\theta$$

We observe that for $R \geq 1$,

$$\left| iR\int_0^{\pi} \frac{e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}e^{i\theta}d\theta \right| \leq R\int_0^{\pi} \frac{e^{-tR\sin\theta}}{R^2-1}d\theta = \frac{R}{R^2-1}\int_0^{\pi} e^{-tR\sin\theta}d\theta \xrightarrow[R\to\infty]{} 0$$

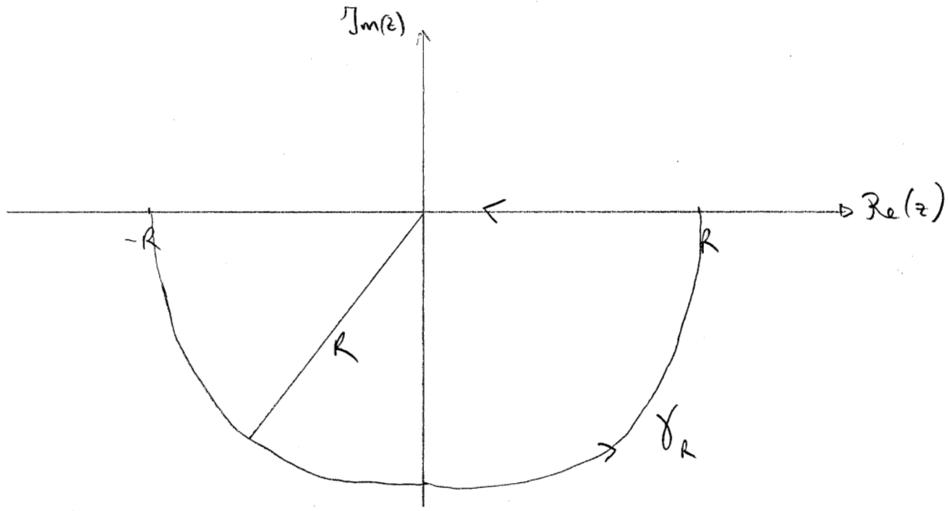

Figure 2: Contour in the complex plane for the second case in Example 1

Hence,

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2} dx = \lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \pi e^{-t} \quad , t \geq 0$$

• <u>If $t \leq 0$</u>, then we adopt the same general strategy, but this time close the integration path by taking the semi-circle in the lower half-plane, as shown in Figure 1. The contribution to the integral along γ_R of the integral along the semi-circle will vanish in the limit $R \to 0$, so

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx = -\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = -2\pi i Res_{z=-i} f(z) = -\frac{2\pi i e^t}{-2i} = \pi e^t \quad , t \leq 0$$

We conclude that $\forall t \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2} dx = \pi e^{-|t|}$$

2 Example 2

In this example, we would like to evaluate the integral

$$I = \int_0^{+\infty} \frac{x^{1/2}}{1+x^2} dx$$

Once again, the improper integral is meaningful in the sense that it converges. To see this, let $h(x) = x^{1/2}/(1+x^2)$. We have

$$h(x) \sim_{x\to 0} \sqrt{x}$$
$$h(x) \sim_{x\to\infty} x^{3/2}$$

The function $\sqrt{x}$ is integrable in 0, and the function $x^{3/2}$ is integrable at $+\infty$, so the integral is well defined. In order to compute it, we let

$$f(z) := \frac{z^{1/2}}{1+z^2}$$

where $z^{1/2}$ is chosen to be the branch of the square root such that $\left(re^{i\theta}\right)^{1/2} = r^{1/2}e^{i\theta/2}$ for $0 \leq \theta < 2pi$. The reason behind the choice of this branch will be apparent shortly.
To compute the desired integral, we consider the "keyhole contour" shown in Figure 2. Let us look at the contributions to the complex integral from the separate elements in the keyhole contour.

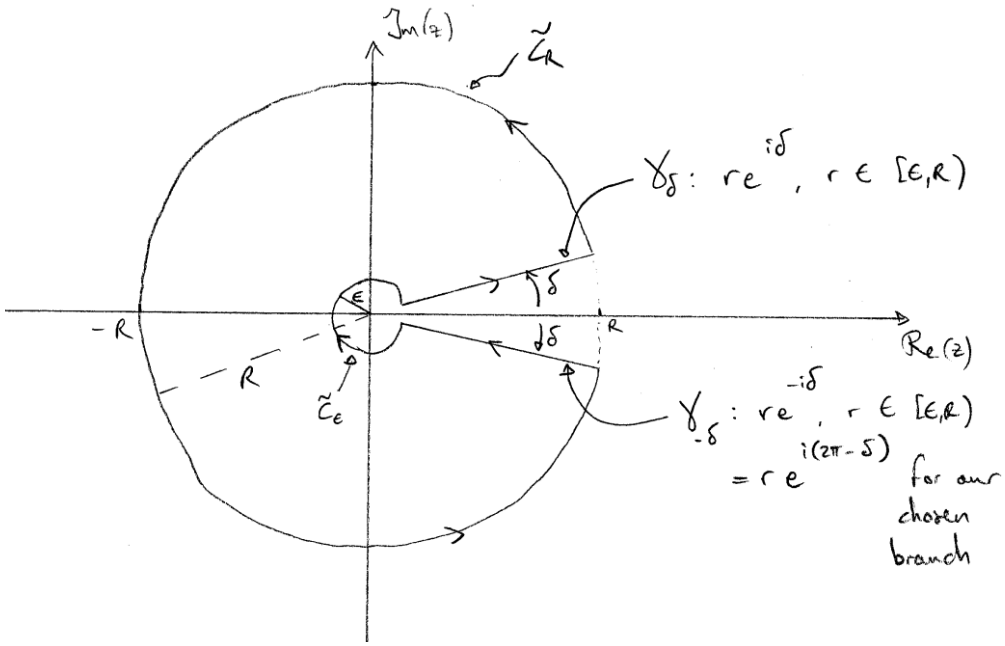

Figure 3: Keyhole contour in the complex plane for the evaluation of the integral in Example 2

- On the large quasi-circle $\tilde{C}_R$ of radius R, we have

$$\forall z \in \tilde{C}_R \ , \ |f(z)| \leq \frac{R^{1/2}}{R^2 - 1} \quad \Rightarrow \quad \int_{\tilde{C}_R} f(z)dz \xrightarrow[R\to\infty]{} 0$$

- On the small quasi-circle $\tilde{C}_\epsilon$ of radius ϵ, we have

$$\forall z \in \tilde{C}_\epsilon \ , \ |f(z)| \leq \frac{\epsilon^{1/2}}{1 - \epsilon^2} \quad \Rightarrow \quad \int_{\tilde{C}_\epsilon} f(z)dz \xrightarrow[\epsilon\to 0]{} 0$$

- On the segment γ_δ, we have

$$\forall z \in \gamma_\delta \ , \ f(z) = \frac{r^{1/2}e^{i\delta/2}}{1 + r^2 e^{2i\delta}} \quad \Rightarrow \quad \int_{\gamma_\delta} f(z)dz = \int_\epsilon^R \frac{r^{1/2}e^{3i\delta/2}}{1 + r^2 e^{2i\delta}} dr$$

And we note that by the dominated convergence theorem

$$\lim_{\delta\to 0} \int_{\gamma_\delta} f(z)dz = \int_\epsilon^R \lim_{\delta\to 0} \frac{r^{1/2}e^{3i\delta/2}}{1 + r^2 e^{2i\delta}} dr = \int_\epsilon^R \frac{r^{1/2}}{1 + r^2} dr$$

where we could use the dominated convergence theorem because for δ small enough, say $\delta \leq \pi/6$, it is clear that

$$\left| \frac{r^{1/2}}{1 + r^2 e^{2i\delta}} \right| \leq \frac{r^{1/2}}{1 + \frac{r^2}{2}}$$

The expression on the right of this inequality is integrable on any interval $[\epsilon, R]$.

- On the segment $\gamma_{-\delta}$, we have

$$\forall z \in \gamma_{-\delta} \ , \ f(z) = \frac{r^{1/2}e^{i\pi}e^{-i\delta/2}}{1 + r^2 e^{-2i\delta}} = -\frac{r^{1/2}e^{-i\delta/2}}{1 + r^2 e^{-2i\delta}} \quad \Rightarrow \quad \int_{\gamma_{-\delta}} f(z)dz = \int_\epsilon^R \frac{r^{1/2}e^{-3i\delta/2}}{1 + r^2 e^{-2i\delta}} dr$$

We are now ready to use the residue theorem, taking the limits $\epsilon \to 0$, $R \to \infty$. We can write

$$2\int_0^{+\infty} \frac{x^{1/2}}{1+x^2}dx = \int_{\gamma_{\text{Key}}} f(z)dz = 2\pi i \left(\text{Res}_{z=i}f(z) + \text{Res}_{z=-i}f(z)\right) = 2\pi i \left(\frac{e^{i\pi/4}}{2i} - \frac{e^{i3\pi/4}}{2i}\right) = \pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \pi\sqrt{2}$$

We conclude that

$$\int_0^{+\infty} \frac{x^{1/2}}{1+x^2}dx = \pi\frac{\sqrt{2}}{2}$$

Reflecting on this derivation, it is now clear why one chose this particular branch of the square root and the corresponding keyhole contour. We picked it in such a way that when one closes the keyhole by taking the limits $\epsilon \to 0$ and $R \to 0$, the remaining contributions, from the pieces corresponding to the branch cut, correspond to the integration domain along the real axis of the integral one is interested in. Here, it was $(0, +\infty)$.

Note: This example was chosen as a typical situation in which constructing a keyhole contour is the appropriate general method to evaluate the improper integral. It turns out, however, that in this particular case, there is perhaps a simpler way to calculate the integral. Indeed, consider the change of variable

$$u^2 = x \quad , \quad 2udu = dx$$

We can rewrite I as

$$I = 2\int_0^{+\infty} \frac{u^2}{1+u^4}du = \int_{-\infty}^{+\infty} \frac{u^2}{1+u^4}du$$

Let

$$g(z) := \frac{z^2}{1+z^4}$$

g has two poles in the upper half plane: $e^{i\pi/4}$ and $e^{3i\pi/4}$, and

$$\text{Res}_{z=e^{i\pi/4}}g(z) = \frac{i}{4e^{i3\pi/4}} = \frac{e^{-i\pi/4}}{4} \quad , \quad \text{Res}_{z=e^{i3\pi/4}}g(z) = \frac{e^{i3\pi/2}}{4e^{i\pi/4}} = \frac{e^{i5\pi/4}}{4}$$

By using a contour γ_R just like in the case $t \geq 0$ of example 1, we find

$$I = 2\pi i \left(-\frac{\sqrt{2}}{4}i\right) = \pi\frac{\sqrt{2}}{2}$$

as expected.

3 Example 3

In this example, we would like to compute the integral

$$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x}dx$$

As before, we first verify that the integral is well defined. There is no issue in $x = 0$ because $\sin x/x$ can be continuously extended there. The potential issue is at the end of the integration domain. Let $(r, R) \in \mathbb{R}^2$ such that $r < R$.

$$\int_r^R \frac{\sin x}{x}dx = \underbrace{\left[-\frac{\cos x}{x}\right]_r^R}_{\xrightarrow[R\to+\infty]{} \frac{\cos r}{r}} - \underbrace{\int_r^R \frac{\cos x}{x^2}dx}_{\text{convergent as } R\to+\infty}$$

We see that the integral is convergent as $R \to +\infty$, and a similar reasoning would be the convergence for $R \to -\infty$. The integral is therefore well defined.
Note that I is an interesting case of an integral which is defined in the Riemann sense, but not in the Lebesgue sense.

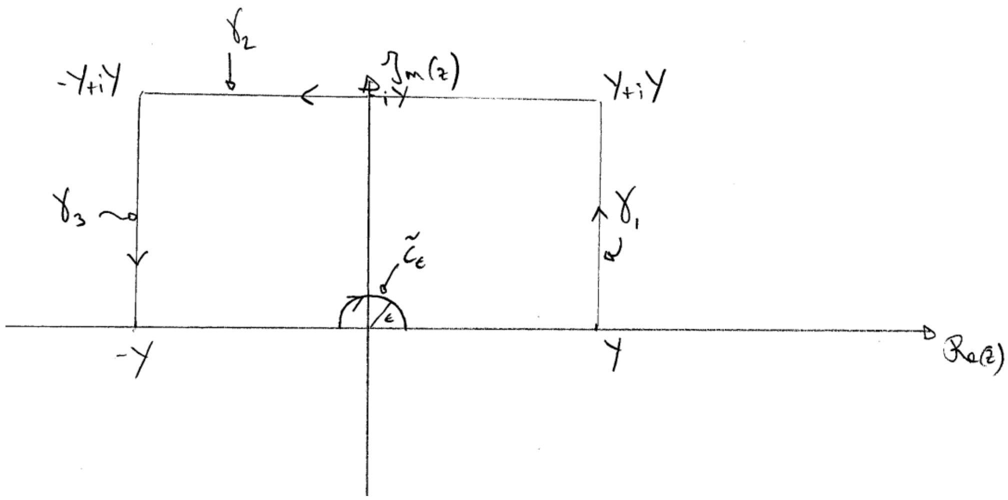

Figure 4: Contour in the complex plane for the evaluation of the integral in Example 3

The idea, as before, is to extend the integration path in the complex plane and use the residue theorem to compute the integral. Intuitively, one wants to introduce the integral

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx$$

There is a difficulty, however: this integral is clearly not defined, because of the nature of the singularity at $x = 0$. What we mean by this integral is the Cauchy principal value, defined by

$$\text{P.V.}\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx\right) := \lim_{\epsilon\to 0}\left(\int_{-\infty}^{\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{+\infty} \frac{e^{ix}}{x}\right) = \underbrace{\text{P.V.}\left(\int_{-\infty}^{+\infty} \frac{\cos x}{x} dx\right)}_{=0 \text{ by parity argument}} + i\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx$$

The contour γ we use for the evaluation of the Cauchy principal value

$$P = \text{P.V.}\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx\right)$$

is shown in Figure 3.
By Cauchy's theorem for simply connected domains, we can write

$$\int_{\gamma} \frac{e^{iz}}{z} dz = 0 = \text{P.V.}\left(\int_{-Y}^{Y} \frac{e^{ix}}{x} dx\right) + \lim_{\epsilon\to 0}\int_{\tilde{C}_\epsilon(0)} \frac{e^{iz}}{z} dz + \int_{\gamma_1} \frac{e^{iz}}{z} dz + \int_{\gamma_2} \frac{e^{iz}}{z} dz + \int_{\gamma_3} \frac{e^{iz}}{z} dz$$

- One can write

$$\frac{e^{iz}}{z} = \frac{1}{z} + g(z)$$

with g analytic in the neighborhood of $z = 0$.
Hence

$$\lim_{\epsilon\to 0}\int_{\tilde{C}_\epsilon(0)} \frac{e^{iz}}{z} dz = \lim_{\epsilon\to 0}\int_{\tilde{C}_\epsilon(0)} \frac{dz}{z} = -\pi i$$

- $$\left|\int_{\gamma_1} \frac{e^{iz}}{z} dz\right| \leq Y\int_0^1 \frac{e^{-tY}}{|Y + itY|} dt \leq \int_0^1 e^{-tY} dt \xrightarrow[Y\to+\infty]{} 0$$

- $$\left|\int_{\gamma_3} \frac{e^{iz}}{z} dz\right| \leq Y\int_1^0 \frac{e^{(t-1)Y}}{|-Y + i(1-t)Y|} dt \leq \int_0^1 e^{(t-1)Y} dt \xrightarrow[Y\to+\infty]{} 0$$

- $$\left|\int_{\gamma_2} \frac{e^{iz}}{z} dz\right| \leq Y \int_{-1}^{1} \frac{e^{-Y}}{|iY - tY|} dt \leq 2e^{-Y} \xrightarrow[Y\to+\infty]{} 0$$

Taking the limit $Y \to +\infty$, we thus find

$$0 = \text{P.V.}\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx\right) - \pi i$$

We therefore conclude that

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi$$

in agreement with the well-known result.

4 Final words

With these three examples, we have learned some fundamental techniques for contour integration, which you will find to be useful for the calculation of many integrals. However, the examples do not cover all the situations you may encounter, and all the types of contours which can make the calculations relatively easy. You are therefore strongly encouraged to practice these techniques with a wide variety of problems, in order to be exposed to all possible situations!