1 The residue theorem

1.1 Residue of a function at a point

Definition: Consider a function f which is analytic in an open connected set Ω except for the isolated singularity at a. Consider a circle C centered in a and contained in Ω. Let

$$P = \int_C f(z)dz$$

If we set $R = \frac{P}{2\pi i}$, the function

$$g(z) := f(z) - \frac{R}{z-a} \quad , \quad \forall z \in \Omega \setminus \{a\}$$

is such that

$$\int_C g(z)dz = 0$$

R as defined above is called the *residue of f at a*:

$$\text{Res}_{z=a} f(z) = \frac{1}{2\pi i} \int_{C_R(a)} f(z)dz$$

Of course, the definition only makes sense if it is independent of the choice of the radius $R > 0$ of the circle C. By Cauchy's theorem, this is not too hard to see.

Consider a second circle $C_{R'}(a)$ centered in a and contained in Ω and the cycle γ made of the piecewise differentiable green, red and black arcs shown in Figure 1. By the general form of Cauchy's theorem,

$$\int_\gamma f(z)dz = 0 \quad \Leftrightarrow \quad \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz + I_\epsilon$$

where I_ϵ is the contribution from the two black horizontal segments separated by a distance ϵ.

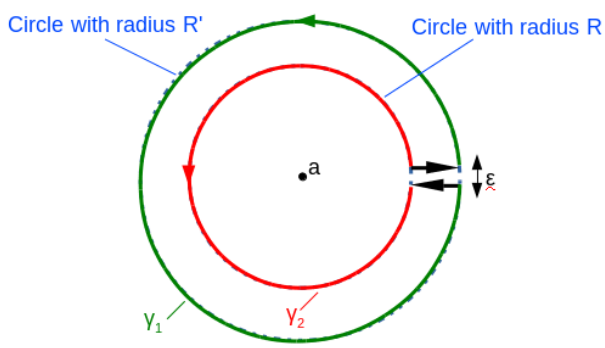

Figure 1: Arcs used to prove that the definition of the residue is independent of the radius of the circle C

Since f is continuous in $\Omega \setminus \{a\}$, $\lim_{\epsilon \to 0} I_\epsilon = 0$, which proves that

$$\int_{C_{R'}(a)} f(z)dz = \int_{C_R(a)} f(z)dz$$

1.2 The residue theorem

Consider a function f which is analytic in the open connected set Ω except for finitely many singularities a_j. Let γ be a cycle in $\Omega' = \Omega \setminus \{a_j\}_{j=1,\ldots,N}$ which is homologous to zero with respect to Ω. Then $\gamma \sim \sum_{j=1}^N n(\gamma, a_j) C_j \pmod{\Omega'}$, where C_j is any circle centered in a_j and contained in Ω'.

By the general formulation of Cauchy's theorem, we can thus write

$$\int_\gamma f(z)dz = \sum_{j=1}^{N} n(\gamma, a_j) \int_{C_j} f(z)dz$$
$$\Leftrightarrow \frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{j=1}^{N} n(\gamma, a_j) \text{Res}_{z=a_j} f(z) \tag{1}$$

The result can naturally be extended to the case in which f has infinitely many singularities, as we have done in Lecture 8. The sum in (1) is always finite, and known as the residue theorem.

Theorem (Residue theorem): Let f be analytic except for isolated singularities a_j in an open connected set Ω. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_j n(\gamma, a_j) \text{Res}_{z=a_j} f(z) \tag{2}$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j. The sum (2) is finite.

1.3 Computing residues

As one may expect, the residue theorem is particularly convenient to use when γ is such that $\forall a_j$, $n(\gamma, a_j) = 0$ or 1.
More importantly, it is only useful as a tool for integration if there is a simple method to compute residues. When f has essential singularities, such a method is not available, and residue calculus is not particularly useful.
However, consider the situation in which f has a pole of order N in a. Then $g(z) = (z - a)^N f(z)$ is analytic in a neighborhood of a. Integrating along a circle C centered in a and in that neighborhood, we may write

$$g^{(N-1)}(a) = \frac{(N-1)!}{2\pi i} \int_C \frac{g(z)}{(z-a)^N} dz = (N-1)! \text{Res}_{z=a} f(z)$$

Hence,

$$\text{Res}_{z=a} f(z) = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left[(z-a)^N f(z) \right] \bigg|_{z=a} \tag{3}$$

In particular, if $f(z) = g(z)/h(z)$ and h has a simple zero at a and $g(a) \neq 0$,

$$\text{Res}_{z=a} f(z) = \frac{g(a)}{h'(a)}$$

Example: Use the residue theorem to compute

$$\oint_{|z|=1} \frac{e^{iz}}{z^2} dz$$

where the circle is traversed in the counterclockwise direction.

2 The argument principle

2.1 The argument principle

Theorem (The argument principle – Part 2): If f is meromorphic in an open connected set Ω, with zeros a_j and poles b_k, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \tag{4}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros and poles. The sums in (4) are finite, and multiple zeros and poles have to be repeated as many times as their order indicates.

Proof: Let us first assume that the function has a finite number of zeros and poles, and call K that number. Consider the orders N_j of the zeros and poles z_j of f in Ω. $N_j > 0$ if z_j is a zero of f, $N_j < 0$ if z_j is a pole of f. Let

$$g(z) := f(z)\Pi_{j=1}^{K}(z - z_j)^{-N_j}$$

g only has removable singularities in Ω, so we can view it as analytic in Ω. Furthermore, g does not have zeros inside Ω. Writing $f(z) = g(z)\Pi_{i=j}^{K}(z - z_j)^{N_j}$ and taking the logarithmic derivative of that equality for $z \neq z_j$, we find

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{K} \frac{N_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

We integrate this equality along any cycle γ homologous to zero in Ω and which does not pass through any z_j. By Cauchy's theorem, $\int_\gamma g'(z)/g(z) = 0$, and by the definition of the index of a point with respect to a curve applied to the remainder of the right-hand side (or by the residue theorem if you prefer to see it this way), we get the desired result □

The proof can be extended to the situation in which the function f may have an infinite number of zeros and/or poles, using the same method as we did in Lecture 8 to show that the formula in (4) remains true, with the sums still finite.

2.2 Rouché's theorem

Rouché's theorem can be viewed as a corollary of the argument principle. It can be stated as follows.

Theorem (Rouché's theorem): Let γ be a cycle which is homologous to zero in the open connected set Ω and such that $n(\gamma, z)$ is either 0 or 1 for all $z \in \Omega$ such that $z \notin \gamma$. Suppose that f and g are analytic in Ω, and that $\forall z \in \gamma$, $|f(z) - g(z)| < |f(z)|$. Then f and g have the same number of zeros *enclosed* by γ.

Proof: From the hypotheses of the theorem, we know that $\forall\, z \in \gamma$, $f(z) \neq 0$ and $g(z) \neq 0$. Along γ, we can therefore consider the function $\psi(z) := g(z)/f(z)$. ψ is such that

$$\forall\, z \in \gamma,\ |\psi(z) - 1| < 1$$

Hence,

$$\int_\gamma \frac{\psi'(z)}{\psi(z)} = \int_\Gamma \frac{d\zeta}{\zeta} = 2\pi i n(\Gamma, 0) = 0$$

where we have used the change of variable $\zeta = \psi(z)$, $\Gamma = \psi(\gamma)$ to derive the first equality.
Now, let N_g be the number of zeros of g inside γ, and N_f the number of zeros of f inside γ. By the argument principle,

$$0 = \int_\gamma \frac{\psi'(z)}{\psi(z)} dz = N_g - N_f \quad \Leftrightarrow \quad N_f = N_g \quad \square$$

Typical example of the use of Rouché's theorem

Consider the polynomial $z^4 - 6z + 3$. How many zeros does it have in the annulus between $|z| = 1$ and $|z| = 2$?

Start with $\gamma_1 :\ |z| = 2$, and take $f_1(z) = z^4$, $g_1(z) = z^4 - 6z + 3$.

$$\forall z \in \gamma_1\ ,\ |f_1(z) - g_1(z)| = |6z - 3| \leq 15 < 16 = |f_1(z)|$$

Hence both f_1 and g_1 have 4 zeros inside $|z| = 2$.

Now consider $\gamma_2 :\ |z| = 1$, and define $f_2(z) = -6z$, $g_2(z) = z^4 - 6z + 3$

$$\forall z \in \gamma_2\ ,\ |f_2(z) - g_2(z)| = |z^4 + 3| \leq 4 < 6 = |f_2(z)|$$

So both f_2 and g_2 have 1 zero inside $|z| = 1$
We conclude that $z^4 - 6z + 3 = 0$ has 3 roots in the annulus.