On several occasions in this course we pointed out close links between results obtained for analytic functions and results concerning harmonic functions we may already know from courses on Partial Differential Equations. The purpose of this lecture is to give these links a rigorous background.

1 Harmonic conjugate

1.1 Harmonic functions

Definition: A function $u \,:\, (x,y) \in \Omega \to \mathbb{R}$ is harmonic if

$$\Delta u \;:= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;\;, \;\; \forall\, (x,y) \in \Omega$$

We already know that if $f(z) = u(x,y) + iv(x,y)$, with $z = x+iy$, is analytic in Ω, then u and v satisfy the Cauchy-Riemann relations, and are therefore harmonic in Ω.

Furthermore, it is easy to see that if u is harmonic in Ω, then $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in Ω, since the real and imaginary parts of g satisfy the Cauchy-Riemann relations.
Thus, the natural question which comes to mind is the following: under which conditions does a harmonic function u on Ω have a harmonic conjugate $v \,:\, \Omega \,\to \mathbb{R}$ such that $f = u + iv$ is analytic on Ω?

To see that the answer to this question is not trivial, condider the function $u(x,y) = \ln(\sqrt{x^2+y^2})$ on the set $\mathbb{R}^2 \setminus \{0\}$. The issue in that particular case is that there does not exist a single-valued conjugate function.

1.2 Conjugate differential of u

As we said before, if u is harmonic in an open connected set Ω, $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in Ω. We may write the differential

$$f dz = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)$$

In the real part, we recognize the differential du. If u has a conjugate harmonic function v, the imaginary part is dv. In general, however, there does not exist a single-valued v. We thus define

$$\star du := -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$\star du$ is called the *conjugate differential* of u, and $\star$ is called the Hodge $\star$-operator. We may write

$$f dz = du + i \star du$$

Lemma: Let γ be a cycle in Ω

$$\int_\gamma \star du = 0 \quad \text{if} \;\; \gamma \sim 0 \;(\text{mod}\; \Omega)$$

The proof is immediate:

$$\int_\gamma f(z)dz = \int_\gamma du + i\int_\gamma \star du$$

The integral on the left-hand side is equal to zero by Cauchy's theorem, since $\gamma \sim 0$ (mod Ω). The first integral on the right-hand size is equal to zero since du is an exact differential and γ is a cycle. We conclude that $\int_\gamma \star du = 0$ if $\gamma \sim 0$ (mod Ω).

Theorem: In a simply connected open set Ω, any harmonic function u has a single-valued conjugate function v which is uniquely determined up to an additive constant.

Proof: For the proof of existence, observe that since Ω is simply connected, $\int_\gamma \star du = 0$ for all cycles γ in Ω. $\star du$ is therefore an exact differential on Ω, and by the theorem at the end of page 3 in Lecture 4, we know that there is a single-valued function $v(x,y)$ such that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad , \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

This v is a single-valued conjugate function of u.

To show uniqueness, consider two such harmonic conjugates v_1 and v_2. $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are both analytic on Ω, so $f_1 - f_2 = i(v_1 - v_2)$ is analytic on Ω. We thus have an analytic function $g = f_1 - f_2$ which maps Ω to the imaginary axis. By the open mapping theorem, g must be a constant: $\exists\, K \in \mathbb{C}$ such that $f_1 = f_2 + K$ $\square$

2 Mean-value property

In what follows, we will often use $(x, y) \in \mathbb{R}^2$ and $z = x + iy \in \mathbb{C}$ interchangeably, and allow ourselves this abuse of notation for the sake of the simplicity of the expressions.

2.1 The mean-value property

Theorem: Consider an open connected set Ω and a harmonic function $u : \Omega \to \mathbb{R}$. Let $\overline{D}_R(z_0) \subset \Omega$ be a closed disk. Then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\theta}) d\theta$$

Proof: From the previous section, we know that u has a harmonic conjugate v on the disk $\overline{D}_R(z_0)$. We can then consider $f = u + iv$, which is analytic, and use the Cauchy integral formula to write

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$

Taking the real part of this equality yields the desired theorem.

Corollary: A nonconstant harmonic function has neither a maximum nor a minimum in any open connected set in which it is defined. Consequently, if a nonconstant harmonic function is defined on a closed bounded set E, its maximum and minimum are taken on the boundary of E.

Proof: Suppose u reaches a maximum M at a point z_0 in the interior of Ω. There exists $R > 0$ such that $D_R(z_0) \subset \Omega$ and $\forall z \in D_R(z_0)$, $u(z) \leq u(z_0)$. Suppose there exists $a \in D_R(z_0)$ such that $u(a) < u(z_0) = M$. Consider the circle with radius r centered in z_0 and going through a, By the mean-value theorem,

$$M = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta < M$$

This is a contradiction.
To obtain the result regarding the minimum, apply the same proof to the harmonic function $w = -u$ $\square$

Corollary: If u_1 and u_2 are two continuous functions on a closed bounded set E which are harmonic in the interior of E and such that $u_1 = u_2$ on the boundary of E, then $u_1 = u_2$ in E.

In other words, functions satisfying the conditions above are uniquely determined by their values on the boundary.

2.2 Poisson's formula

Theorem: Suppose that u is harmonic on $D_R(0)$ and continuous on $\overline{D}_R(0)$. Then, $\forall a \in D_R(0)$,

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z) d\theta \tag{1}$$

Proof: The idea is that the mean-value theorem gives us a formula for $u(0)$. All we need to do is apply a linear transformation which moves the point a to the center. This is done as follows.
Let us assume first that u is harmonic on $\overline{D}_R(0)$. The linear transformation

$$S : \zeta \mapsto z = R\frac{R\zeta + a}{R + \overline{a}\zeta}$$

maps $\overline{D}_1(0)$ onto $\overline{D}_R(0)$, and is such that $S(0) = a$.
Now, the function $\zeta \in D_1(0) \mapsto u(S(\zeta))$ is harmonic on $\overline{D}_1(0)$. By the mean value property, we can write

$$u(S(0)) = u(a) = -\frac{i}{2\pi}\int_{|\zeta|=1} u(S(\zeta))\frac{d\zeta}{\zeta}$$

We know that S can be inverted, with inverse

$$\zeta = S^{-1}(z) = \frac{R(z-a)}{R^2 - \overline{a}z}$$

Hence,

$$\frac{d\zeta}{\zeta} = \left(\frac{1}{z-a} + \frac{\overline{a}}{R^2 - \overline{a}z}\right)dz$$

When $|\zeta| = 1$, $|z| = R$, so introducing the argument θ of z on $|z| = R$, we have

$$\frac{d\zeta}{\zeta} = \left(\frac{iz}{z-a} + \frac{iz\overline{a}}{R^2 - \overline{a}z}\right)d\theta = \left(\frac{iz}{z-a} + \frac{i\overline{a}}{\overline{z} - \overline{a}}\right) = i\left(\frac{R^2 - |a|^2}{|z-a|^2}\right)d\theta$$

so that

$$u(a) = \frac{1}{2\pi}\int_{|z|=R}\frac{R^2 - |a|^2}{|z-a|^2}u(z)d\theta$$

as desired.

This formula is known as Poisson's formula. We may get an alternate form for it by observing that

$$\frac{R^2 - |a|^2}{|z-a|^2} = \Re\left(\frac{R^2 - |a|^2}{|z-a|^2}\right) = \Re\left(\frac{z}{z-a} + \frac{\overline{a}}{\overline{z} - \overline{a}}\right) = \frac{1}{2}\left(\frac{z+a}{z-a} + \frac{\overline{z}+\overline{a}}{\overline{z}-\overline{a}}\right) = \Re\left(\frac{z+a}{z-a}\right)$$

We have shown that Poisson's formula could also be written as

$$u(a) = \frac{1}{2\pi}\int_{|z|=R}\Re\left(\frac{z+a}{z-a}\right)u(z)d\theta$$

which may yet again be rewritten as follows:

$$\forall a \in D_R(0)\ ,\ u(a) = \Re\left(\frac{1}{2\pi i}\int_{|z|=R}\frac{z+a}{z-a}\frac{u(z)}{z}dz\right) \tag{2}$$

The function in parenthesis in (2) is an analytic function of a for $|a| < R$, so the expression above gives us the conjugate function of u: u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}\frac{u(\zeta)}{\zeta}d\zeta + iK \quad , \quad K \in \mathbb{R} \tag{3}$$

This result is sometimes called Schwarz's formula.

The careful reader will have noticed that we still have not proved that Poisson's formula holds even if u is harmonic in the *open* disk only, and continuous in the closed disk.
To address that case, consider δ such that $0 < \delta < 1$ and $\tilde{u} := u(\delta z)$.
$\tilde{u}$ is harmonic in $\overline{D}_R(0)$ so we can write $\forall a \in D_R(0)$,

$$u(\delta a) = \frac{1}{2\pi}\int_{|z|=R}\frac{R^2 - |a|^2}{|z-a|^2}u(\delta z)d\theta$$

Now, u is continuous on the compact set $\overline{D}_R(0)$, so it is uniformly continuous on that set by the Heine-Cantor theorem. Taking the limit $\delta \to 1$ in the modified Poisson's formula above, we find by uniform continuity that Poisson's formula holds for the closed disk as well.

2.3 Schwarz's theorem

Poisson's formula can be viewed as a way to define a function inside a disk from the values $u(z)$ on the circle $|z| = R$ of a function u which may only be defined on that circle. We have seen that a function defined in this way is harmonic, as the real part of an analytic function. A natural question then is: does this function have boundary value $u(z)$ on $|z| = R$?
Schwarz's theorem, given below, answers this question.

Theorem (Schwarz's theorem): Given a piecewise continuous function u on $[0, 2\pi]$, the Poisson integral

$$P_u(z) = \frac{1}{2\pi} \int_0^{2\pi} \Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) u(\theta) d\theta$$

is harmonic for $|z| < 1$, and $\lim_{z \to e^{i\theta_0}} P_u(z) = u(\theta_0)$ if u is continuous at θ_0.

Proof: We already know that P_u is harmonic for $|z| < 1$.

• Now, P is a linear operator which maps piecewise continuous functions u on $[0, 2\pi]$ to harmonic functions P_u on the open disk. Indeed, it is easy to show that $P_{u_1+u_2} = P_{u_1} + P_{u_2}$, and that $P_{cu} = cP_u$ for $c \in \mathbb{R}$ constant.

Applying Poisson's formula to $u = 1$, we find by direct calculation that $P_1 = 1$, and thus $P_c = c$, $\forall c \in \mathbb{R}$.
Finally, if $u \geq 0$ on $[0, 2\pi]$, $P_u \geq 0$.

Combining all these properties, we conclude that if there exists $(m, M) \in \mathbb{R}^2$ such that $m \leq u(\theta) \leq M$ $\forall \theta \in [0, 2\pi]$, then $m \leq P_u \leq M$

• Without loss of generality, assume $u(\theta_0) = 0$ (if that does not hold, consider $u - u(\theta_0)$ instead, which we can do without difficulty because $P_c = c$). Then take a short arc $C_2 \subset \partial D_1(0)$ with $e^{i\theta_0}$ in its interior. By continuity of u in θ_0, $\forall \epsilon > 0$, one can choose C_2 small enough that one can say $|u(\theta)| < \epsilon/2$ $\forall \theta$ such that $e^{i\theta} \in C_2$. Consider the complement C_1 of C_2 in $\partial D_1(0)$, and define u_1 and u_2 such that

$$u_1(\theta) = \begin{cases} u(\theta) & \text{for } \theta \text{ such that } e^{i\theta} \in C_1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(\theta) = \begin{cases} u(\theta) & \text{for } \theta \text{ such that } e^{i\theta} \in C_2 \\ 0 & \text{otherwise} \end{cases}$$

By linearity of P, we have $P_u = P_{u_1} + P_{u_2}$

$|u_2(\theta)| < \epsilon/2$ $\forall \theta \in [0, 2\pi]$ so $|P_{u_2}(z)| < \epsilon/2$ $\forall z \in D_1(0)$.
Furthermore, P_{u_1} is harmonic everywhere except on C_1. P_{u_1} is zero on C_2, and by continuity $P_{u_1}(z) \to 0$ as $z \to e^{i\theta_0} \in C_2$. Hence, $forall \epsilon$, $\exists \delta$ such that $|z - e^{i\theta_0}| < \delta \Rightarrow |P_{u_1}(z)| < \epsilon/2$.

We conclude that for $|z| < 1$ such that $|z - e^{i\theta_0}| < \delta$,

$$|P_u(z)| \leq |P_{u_1}(z)| + |P_{u_2}(z)| = \epsilon$$

Since ϵ is arbitrary, this completes our proof □

3 Schwarz reflection principle

The idea of the Schwarz reflection principle is to extend an analytic function $f : \Omega \to \mathbb{C}$ to a larger domain, with the ultimate goal to find the maximal domain on which f can be defined and analytic.

We start by remembering that if f is analytic on Ω, then $\overline{f(\overline{z})}$ is analytic on $\tilde{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$ (see Problem Set 1).
Now, if f is an analytic function defined on an open connected set Ω which is symmetric about the x-axis, and $f(z) = \overline{f(\overline{z})}$, then f is real on the intersection of the x-axis with Ω. We have the following converse:

Theorem: Let Ω be an open connected set which is symmetric with respect to the x-axis, and let $\Omega^+ = \Omega \cap \{\Im(z) > 0\}$, $\sigma = \Omega \cap \{\Im(z) = 0\}$. If f is continuous on $\Omega^+ \cup \sigma$, analytic on Ω^+, and real for all $z \in \sigma$, then f has an analytic continuation to all of Ω such that $f(z) = \overline{f(\overline{z})}$.

The theorem above follows from the following theorem regarding harmonic functions, which we will prove first:

Theorem: Suppose v is continuous on $\Omega^+ \cup \sigma$, harmonic on Ω^+, and zero on σ. Then v has a harmonic extension to Ω satisfying $v(z) = -v(\overline{z})$.

Proof: Let $\Omega^- = \Omega \cap \{\Im(z) < 0\}$. We construct the function V on Ω such that $V(z) = v(z)\ \forall z \in \Omega^+$, $V(z) = 0\ \forall z \in \sigma$, and $V(z) = -v(\overline{z})\ \forall z \in \Omega^-$. Let us show that V is harmonic in Ω, i.e. that V is harmonic on σ.

$\forall z_0 \in \sigma$, consider δ small enough that $\overline{D}_\delta(z_0) \subset \Omega$. Let P_V be the Poisson integral of V with respect to $\partial D_\delta(z_0)$. By Schwarz's theorem, P_V is harmonic on $D_\delta(z_0)$ and continuous on $\overline{D}_\delta(z_0)$. In the upper half disk, $V - P_V$ is harmonic. Furthermore, $V - P_V = 0$ on the upper semi-circle, again by Schwarz's theorem. Now, on $\sigma \cap D_\delta(z_0)$, $V = 0$ by construction, and

$$P_V(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re(\frac{\delta e^{i\theta} + z}{\delta e^{i\theta} - z}) U(\delta e^{i\theta}) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta^2 - |z|^2}{(\delta e^{i\theta} - z)(\delta e^{-i\theta} - z)} V(\delta e^{i\theta}) = 0$$

where the last equality follows from a symmetry argument for the integrand.
Applying the maximum and minimum principle to $V - P_V$, we conclude that $V = P_V$ in the upper half disk. With a parallel proof, we can show that $V = P_V$ in the lower half-disk. Hence, $V = P_V$ in the whole disk $D_\delta(z_0)$, and V as constructed above is harmonic in z_0. Since z_0 is an arbitrary point in σ, V is harmonic on σ.

Now that we know how to extend v to a harmonic function V on all of Ω, we can prove the first theorem.

Consider $f = u + iv$ defined on Ω^+. We want to verify that the extension of f defined by $f(z) = \overline{f(\overline{z})} = u(\overline{z}) - iv(\overline{z})$ is indeed an appropriate analytic extension of f on all of Ω.
Let us take $z_0 \in \sigma$, and $D_\delta(z_0)$ as before. We already know from the proof above how v can be extended to a harmonic function V on all of $D_\delta(z_0)$. Now, since the harmonic conjugate of v on $D_\delta(z_0) \cap \{\Im(z) > 0\}$ is $-u$, we construct the harmonic conjugate $-U$ of V. We can always choose U such that $\forall z \in D_\delta(z_0) \cap \{\Im(z) > 0\}$, $U(z) = u(z)$, by adjusting the free constant properly.

Now, consider the function $g(z) := U(z) - U(\overline{z})$. $\forall z \in \sigma$, $g(z) = 0$. Hence $\partial g/\partial x(z) = 0$.
Furthermore, $\partial g/\partial y(z) = 2\partial U/\partial y = -2\partial V/\partial x(z) = 0\ \forall z \in \sigma$.
$h(z) := \partial g/\partial x - i\partial g/\partial y$ is such that $h(z) = 0\ \forall z \in \sigma$. Since g is harmonic on $D_\delta(z_0)$, h is analytic on this set. So we conclude that $h \equiv 0$ in this set. In other words, $\forall z \in D_\delta(z_0)$, $U(z) = U(\overline{z})$. And $f(z) = \overline{f(\overline{z})}$ is an appropriate analytic continuation of f on all of $D_\delta(z_0)$.

To extend the definition on all of Ω, the idea is to repeat the construction on arbitrary disks, recognizing that U coincides on overlapping disks. This part of the proof is left for the reader.