1 Generalizing results about infinite sequences and infinite series

1.1 Weierstrass' theorem

Theorem: Consider the sequence $(f_n)_{n\in\mathbb{N}}$, where f_n is analytic on the open connected set Ω_n. Suppose in addition that $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots$ and that $\cup_{n\in\mathbb{N}} = \Omega$.
If $(f_n)_{n\in\mathbb{N}}$ converges to a limit function f in the open connected set Ω, uniformly on every compact subset of Ω, then f is analytic in Ω. Moreover, f_n' converges uniformly to f' on every compact subset of Ω.

Proof: First observe that every compact subset $E \subset \Omega$ is covered by $(\Omega_n)_{n\in\mathbb{N}}$. Hence $\exists\ N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N},\ n \geq N\ \Rightarrow\ E \subset \Omega_n$.
Let us take $z_0 \in \Omega$ and $R > 0$ such that $\overline{D}_R(z_0) \subset \Omega$. Let us take N such that $\forall\ n \in \mathbb{N},\ n \geq N\ \Rightarrow\ \overline{D}_R(z_0) \subset \Omega_n$. By Cauchy's integral formula, $\forall n \in \mathbb{N}$ such that $n \geq N$,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}_R(z_0)} \frac{f_n(\zeta)}{\zeta - z}d\zeta \ \ , \ \ \forall z \in D_R(z_0)$$

Since f_n converges uniformly to f on $\overline{D}_R(z_0)$, taking the limit $n \to +\infty$ yields

$$f(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta \ \ , \ \ \forall z \in D_R(z_0)$$

which shows that f is analytic in the disk, from which we conclude that f is analytic in Ω.
Furthermore,

$$f_n'(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}_R(z_0)} \frac{f_n(\zeta)}{(\zeta - z)^2}d\zeta \ \ , \ \ \forall z \in D_R(z_0)$$

so

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta = f'(z) \ \ , \ \ \forall z \in D_R(z_0)$$

Uniform convergence of f_n' to f' is straightforward to establish from the fact that $\forall\ n \in \mathbb{N}$ such that $n \geq N$

$$|f_n'(z) - f'(z)| \leq \frac{1}{2\pi}\int_{\partial\overline{D}_R(z_0)} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2}|d\zeta|$$

(simple estimates for the integrand above then yield the desired result) and the fact that any compact subset can be covered by a finite number of closed disks as above.

Note that it is a simple task to use the theorem to show that with the same hypotheses, $\forall k \in \mathbb{N}$ $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of Ω.

Corollary (Hurwitz's Theorem): If the functions f_n are analytic and nowhere zero in an open connected set Ω, and if f_n converges to f uniformly on every compact subset of Ω, then f is either identically zero, or never equal to zero in Ω.

Proof: Suppose there exists $z_0 \in \Omega$ such that $f(z_0) = 0$ and that f is not identically zero. Since f is analytic, $\exists\delta > 0$ such that $\forall z \in \overline{D}_\delta(z_0) \setminus \{z_0\} \subset \Omega$, $f(z) \neq 0$. On $\partial\overline{D}_\delta(z_0)$,

$$\frac{1}{f_n} \xrightarrow[n\to+\infty]{} \frac{1}{f} \quad , \quad f_n' \xrightarrow[n\to+\infty]{} f'$$

both uniformly. We may thus write

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial\overline{D}_\delta(z_0)} \frac{f_n'(z)}{f_n(z)}dz = \frac{1}{2\pi i}\int_{\partial\overline{D}_\delta(z_0)} \frac{f'(z)}{f(z)}dz$$

By the argument principle, the left-hand side is 0, and the equality cannot hold since $f(z_0) = 0$. There is a contradiction. This concludes our proof.

1.2 Infinite series

Another corollary of Weierstrass' theorem: If an infinite series of the form

$$f(z) = f_1(z) + f_2(z) + \ldots + f_n(z) + \ldots$$

where the f_n are analytic on an open connected set Ω, converges uniformly on every compact subset of Ω, then f is analytic in Ω and the series can be differentiated term by term.

This corollary is a direct application of Weierstrass' theorem for the sequence of functions $(g_n)_{n\in\mathbb{N}}$ with $g_n = \sum_{i=1}^{n} f_i$.

Application to a result we already know: the existence of a power series expansion for every analytic function

We will propose a different proof, relying on Weierstrass' theorem and its corollary, of the existence of a power series expansion for every analytic function. Consider a function f which is analytic in the open connected set Ω. For $z_0 \in \Omega$, we know from Lecture 7 that we can write the Taylor formula:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

with

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)} d\zeta$$

and $\overline{D}$ any disk $|z - z_0| \leq R$ contained in Ω.
Let M be the maximum of $|f|$ on $\partial\overline{D}$. We obtain the estimate

$$|f_{n+1}(z)(z - z_0)^{n+1}| \leq \frac{M|z - z_0|^{n+1}}{R^n(R - |z - z_0|)}$$

From this upper bound, we conclude that $f_{n+1}(z)(z - z_0)^{n+1}$ converges uniformly to zero in every disk $|z - z_0| \leq r < R$, which proves the existence of a Taylor series for f centered in z_0 which is valid in the largest open disk of center z_0 contained in Ω.

2 Laurent series

2.1 Analytic functions on an annulus

Let $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ be an annulus. For each analytic function $f : A \to \mathbb{C}$ there are analytic functions $F_1 : \{z \in \mathbb{C} : |z| > R_1\} \to \mathbb{C}$ and $F_2 : \{z \in \mathbb{C} : |z| < R_2\} \to \mathbb{C}$ such that $\forall z \in A$, $f(z) = F_2(z) - F_1(z)$.

Proof: Let $z_0 \in A$, and define for all $z \in A \setminus \{z_0\}$

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

g can be extended to an analytic function on all of A. Let R such that $R_1 < R < R_2$, and the circle $C_R(0)$ with radius R centered in 0. As we have already shown,

$$U = \frac{1}{2\pi i}\int_{C_R(0)} g(z) dz$$

is independent of R, and such that

$$U = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(z)}{z - z_0} dz - f(z_0) n(C_R(0), z_0)$$

We may thus write

$$\begin{cases} U = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(z)}{z - z_0} dz \quad , \quad R_1 < R < |z_0| \\ U = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(z)}{z - z_0} dz - f(z_0) \quad , \quad |z_0| < R < R_2 \end{cases} \tag{1}$$

We define, for R such that $R_1 < R < |z_0|$

$$F_1(z_0) = \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(z)}{z - z_0} dz$$

F_1 is analytic on $\{z_0 : R < |z_0|\}$, and its value is independent of $R \in (R_1, |z_0|)$, so it is an analytic function on $\{z_0 : R_1 < |z_0|\}$.
Likewise, for R such that $|z_0| < R < R_2$, we define

$$F_2(z_0) = \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(z)}{z - z_0} dz$$

which is analytic on $\{z_0 : |z_0| < R_2\}$, following an argument which parallels the argument given just above.
Finally, from (1), we find that $f(z_0) = F_2(z_0) - F_1(z_0)$ □

Observe that the proof and result given above can be generalized to an annulus centered in $a \in \mathbb{C}$ instead of just 0 without difficulty. We will use this fact in the next section.

2.2 Laurent series

Any analytic function f on $A = \{z \in \mathbb{C} : R_1 < |z - a| < R_2\}$ can be developed in a power series of the form

$$f(z) = \sum_{n=-\infty}^{n=+\infty} c_n (z - a)^n \tag{2}$$

The series above, called a Laurent series, converges *uniformly* on A. Moreover,

$$\forall n \in \mathbb{Z} \ , \ n(\gamma, a) c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz$$

Proof: Without loss of generality, we translate A so that $a = 0$. From our previous result, we know that $\forall z \in A$, $f(z) = F_2(z) - F_1(z)$, with F_1 and F_2 as given above.
F_2 is analytic on the disk $|z| < R_2$, so it has a power series

$$F_2(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges uniformly on this disk.
For F_1, we consider the function $G(z) := F_1(1/z)$. From the integral representation of F_1, it is clear that $F_1(z) \to 0$ as $z \to \infty$. Therefore, $G(z) \to 0$ as $z \to 0$, so G can be viewed as an analytic function on the disk $|z| < 1/R_1$. On that disk, one can write $G(z) = \sum_{n=1}^{\infty} b_n z^n$, and the series converges uniformly on the disk.
In other words, the series

$$F_1(z) = \sum_{n=1}^{\infty} b_n z^{-n}$$

converges uniformly on $\{z : |z| > R_1\}$.
We conclude that $\forall z \in A$, $f_z = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n z^{-n}$, with each sum converging uniformly on A.

Using the uniform convergence, we may then write:

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz = \sum_{k=-\infty}^{+\infty} \frac{c_k}{2\pi i} \int_\gamma (z - a)^{k-n-1} dz$$

The integrals on the right-hand side are zero, except when $k - n - 1 = -1$. Hence

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz = c_n n(\gamma, a) \qquad \square$$

2.3 Laurent series for isolated singularities

Let f be holomorphic on $\Omega \setminus \{a\}$, where Ω is an open connected set, and a an isolated singularity. Its Laurent series on the punctured disk $0 < |z-a| < R$ is given by $f(z) = \sum_{n=-\infty}^{n=+\infty} c_n (z-a)^n$.

- f has a removable singularity at a iff $c_n = 0$ for all n such that $n < 0$.
- f has a pole of order N at a iff $c_n = 0$ for all $n < -N$ and $c_{-N} \neq 0$.
- f has an essential singularity at a iff $c_n \neq 0$ for infinitely many negative values of n.

The proof of this last point is left for the reader.

3 Partial fractions and factorization

3.1 Partial fractions

Consider a meromorphic function f in an open connected set Ω, with poles ζ_k. To each pole ζ_k corresponds a singular part of f at ζ_k, as we have seen in Lecture 7, which may be written as $P_k\left(\frac{1}{z-\zeta_k}\right)$, where P_k is a polynomial. It may thus seem like every meromorphic function can be expanded as

$$f(z) = \sum_k P_k\left(\frac{1}{z-\zeta_k}\right) + g(z)$$

where g is analytic in Ω. This is not actually so, because the sum above is infinite in general, and may not be convergent. That being said, under certain conditions and by a clever subtraction of analytic functions, it is often possible to obtain an expansion as desirable as the one above. This is the point of the Mittag-Leffler theorem.

Theorem (Mittag-Leffler theorem): Let $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $\lim_{k\to\infty} = \infty$ and P_k be polynomials without constant term. Then there exist functions f meromorphic in $\mathbb{C}$ with poles at just the points ζ_k and corresponding singular parts $P_k\left(\frac{1}{z-\zeta_k}\right)$.
The most general f of this kind can be written

$$f(z) = g(z) + \sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] \tag{3}$$

where g is analytic and the p_k are polynomials.

Proof: Without loss of generality, we assume $\zeta_k \neq 0$ for all k. Consider the Taylor formula for $P_k\left(\frac{1}{z-\zeta_k}\right)$ around $z=0$. If we call $\psi(z) := P_k\left(\frac{1}{z-\zeta_k}\right)$, we may write

$$\psi(z) = \psi(0) + \psi'(0)z + \frac{\psi''(0)}{2}z^2 + \ldots + \frac{\psi^{(N_k)}(0)}{N_k!}z^{N_k} + \psi_{N_k+1}z^{N_k+1}$$

for an N_k to be specified shortly, and

$$\psi_{N_k+1}(z) = \frac{1}{2\pi i}\int_C \frac{\psi(\zeta)}{\zeta^{N_k+1}(\zeta - z)}d\zeta$$

If we take C to be the circle with radius $|\zeta_k|/2$ and center 0, and consider the maximum M_k of ψ on C, we obtain the upper bound

$$|\psi_{N_k+1}(z)| \leq \frac{1}{2\pi}\frac{2\pi|\zeta_k|}{2}\frac{M_k}{\left(\frac{|\zeta_k|}{2}\right)^{N_k+1}\frac{|\zeta_k|}{4}}$$

for all z such that $|z| \leq |\zeta_k|/4$.

Now, let p_k be the partial sum of ψ up to z_k^N.

$$\forall z : |z| \leq \frac{|\zeta_k|}{4} \ , \ |\psi(z) - p_k(z)| \leq 2M_k \left(\frac{2|z|}{|\zeta_k|}\right)^{N_k+1}$$

Let us choose N_k large enough that $M_k 2^k \leq 2^{N_k}$. Then

$$|\psi(z) - p_k(z)| \leq 2^{-k} \quad \Leftrightarrow \ |P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)| \leq 2^{-k} \ \ , \forall \ z \ : \ |z| \leq \frac{|\zeta_k|}{4} \tag{4}$$

This is the result we need to claim that the sum in the theorem converges uniformly in each disk $\overline{D}_R(0)$ (except at the poles) and thus represents a meromorphic function h.
We write

$$\sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] = \sum_{|\zeta_k|/4 \leq R} \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] + \sum_{|\zeta_k|/4 > R} \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right]$$

Because of Eq.(4), the second term on the right-hand side is analytic in $\overline{D}_R(0)$ by Weierstrass' theorem. The first term on the right-hand side corresponds to a finite sum and has $P_k\left(\frac{1}{z-\zeta_k}\right)$ as the singular part at the pole ζ_k. This proves the existence part of the theorem:

$$h(z) := \sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right]$$

is a meromorphic function.
Finally, it is clear that if f is a meromorphic function with the same poles ζ_k and same singular parts as h, $g = f - h$ is analytic $\square$

3.2 Infinite products of functions

<u>**Definition**</u>: An infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers converges if there exists $N \geq 1$ such that the limit of partial products

$$a = \lim_{n\to\infty} \prod_{k=N}^{n} a_k$$

exists and is nonzero. In this case,

$$\prod_{n=1}^{\infty} a_n := \left(\prod_{n=1}^{N-1} a_n\right) a$$

Let us mention that one may sometimes relax the nonzero part of the definition, and say that $\prod_{n=1}^{\infty} a_n$ converges if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite nonzero limit.

<u>Note</u>: If $\prod_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 1$
One therefore often considers products written in the form $\prod_{n=1}^{\infty}(1 + a_n)$, with $\lim_{n\to\infty} a_n = 0$ as the necessary condition.

Theorem: The infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ with $(1 + a_n \neq 0)$ converges if and only if $\sum_{n=1}^{\infty} \mathrm{Ln}(1 + a_n)$ converges, where Ln is the principal branch of the logarithm.

Proof: • The sufficient condition is straightforward. Let

$$S_n = \sum_{k=1}^{n} \mathrm{Ln}(1 + a_k)$$

We have

$$P_n := \prod_{k=1}^{n}(1 + a_k) = e^{S_n}$$

Hence,

$$S_n \xrightarrow[n\to+\infty]{} S \quad \Rightarrow \quad P_n \xrightarrow[n\to+\infty]{} P = e^S \neq 0$$

• Conversely, let us assume $P_n \xrightarrow[n\to+\infty]{} P \neq 0$
There exists $M_n \in \mathbb{Z}$ such that

$$\text{Ln}\left(\frac{P_n}{P}\right) = S_n - \text{Ln}P + 2\pi i M_n$$

There also exists $M_{n+1} \in \mathbb{Z}$ such that

$$\text{Ln}\left(\frac{P_{n+1}}{P}\right) = S_{n+1} - \text{Ln}P + 2\pi i M_{n+1}$$

Thus,

$$2\pi i(M_{n+1} - M_n) = \text{Ln}\left(\frac{P_{n+1}}{P}\right) - \text{Ln}\left(\frac{P_n}{P}\right) - \text{Ln}(1 + a_{n+1})$$

so that

$$2\pi(M_{n+1} - M_n) = \text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) - \text{Arg}(1 + a_{n+1})$$

Now, $P_n/P \xrightarrow[n\to+\infty]{} 1$, so

$$\text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) \xrightarrow[n\to+\infty]{} 0$$

And since $|\text{Arg}(1+a_{n+1})| \leq \pi$, for n large enough we must have $M_{n+1} = M_n$. In other words, for n sufficiently large, $M_n = M \in \mathbb{Z}$, so for such large n

$$\text{Ln}\left(\frac{P_n}{P}\right) = S_n - \text{Ln}P + 2\pi i M$$

and since $P_n/P \xrightarrow[n\to+\infty]{} 1$,

$$S_n \xrightarrow[n\to+\infty]{} \text{Ln}P - 2\pi i M$$

where the right hand side is now independent of n, as desired. This proves the necessary condition □

We have therefore shown that the question of the convergence of infinite products can be reduced to the well-known question of the convergence of infinite sums. The following theorem turns this result in a simpler form yet. First, we need to introduce the concept of absolute convergence for an infinite product.

Definition: The infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ is said to be absolutely convergent if the infinite sum $\sum_{n=1}^{\infty} \text{Ln}(1 + a_n)$ is absolutely convergent.

Theorem: The product $\prod_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent iff $\sum_{n=1}^{\infty} |a_n|$ converges.

Proof: Convergence for either sum implies $a_n \xrightarrow[n\to+\infty]{} 0$. Hence, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad \frac{1}{2}|a_n| < |\text{Ln}(1 + a_n)| < \frac{3}{2}|a_n|$$

where we have used the fact that

$$\frac{\ln(1 + z)}{z} \xrightarrow[z\to 0]{} 1$$

Therefore

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} |\text{Ln}(1 + a_n)| \text{ converges} \quad \square$$

3.3 Canonical products

If g is an entire function, $f(z) := e^{g(z)}$ is entire and everywhere nonzero. Conversely, let us consider an entire, nonzero function f. Then f'/f is an entire function which has a primitive g which is also entire.

$$\frac{d}{dz}\left[f(z)e^{g(z)}\right] = f'(z)e^{g(z)} - f(z)\frac{f'(z)}{f(z)}e^{-g(z)} = 0 \;\Rightarrow\; f(z)e^{-g(z)} = C \in \mathbb{C}$$

In other words, to within a redefinition of g, we showed that there exists g entire such that $f(z) = e^{g(z)}$.

This result gives us a way to construct the most general entire function with a finite number of zeros. Say f is entire with a zero of order M at the origin, and N zeros a_1, a_2, ..., a_N away from the origin (multiple zeros being repeated). Then

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{N} (1 - \frac{z}{a_n})$$

with g entire.

The previous result suggests the following question: can one construct a similar representation if f has infinitely many zeros?

If $\prod_{n=1}^{\infty}(1 - z/a_n)$ converges uniformly on every compact set of $\mathbb{C}$, then that product represents an entire function with the same zeros as f ($z = 0$ excluded). It is then clear that one can write

$$\frac{f(z)}{\prod_{n=1}^{\infty}(1 - z/a_n)} = z^M e^{g(z)}$$

with g entire.
We see that convergence depends on the zeros a_n of f. From what we have seen before, $\prod_{n=1}^{\infty}(1 - z/a_n)$ converges absolutely iff $\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges absolutely, and convergence is then uniform on every closed disk $|z| \leq R$.
For general $(a_n)_{n\in\mathbb{N}}$ however, we may need to help convergence of the infinite product with multiplying factors, just like we need to help convergence of the infinite sum with additive terms in the Mittag-Leffler case. This is the point of the Weierstrass factorization theorem below.

Theorem (Weierstrass factorization theorem): There exists an entire function with arbitrarily prescribed zeros $(a_n)_{n\in\mathbb{N}}$, as long as $a_n \to \infty$ if the numbers of zeros is infinite. Moreover, every entire function with these and no other zeros can be written as

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{N_n}\left(\frac{z}{a_n}\right)^{N_n}}$$

where the product is taken over all nonzero a_n, the N_n are integers, and g is an entire function.

We want to show that given an arbitrary sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$, with $a_n \xrightarrow[n\to+\infty]{} \infty$, there exist polynomials $p_n(z)$ such that $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)e^{p_n(z)}$ converges to an entire function.
All we have to study is the convergence of

$$\sum_{n=1}^{\infty}\left[\mathrm{Ln}\left(1 - \frac{z}{a_n}\right) + p_n(z)\right] := \sum_{n=1}^{\infty}\psi_n(z)$$

As we did for the proof of the Mittag-Leffler theorem, for $R > 0$ given, we only need to consider the terms with $|a_n| > R$. For $z \in \overline{D}_R(0)$, we may write

$$\ln\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \ldots$$

It is then natural to define

$$p_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \frac{1}{3}\left(\frac{z}{a_n}\right)^3 + \ldots + \frac{1}{N_n}\left(\frac{z}{a_n}\right)^{N_n}$$

for an $N_n \in \mathbb{N}$ to be specified later.
For $z \in \overline{D}_R(0)$,

$$\psi_n(z) = -\frac{1}{N_n+1}\left(\frac{z}{a_n}\right)^{N_n+1} - \frac{1}{N_n+2}\left(\frac{z}{a_n}\right)^{N_n+2} - \dots$$

Hence,

$$|\psi_n(z)| \leq \frac{1}{N_n+1}\left(\frac{R}{|a_n|}\right)^{N_n+1} \frac{1}{1-\frac{R}{|a_n|}} \tag{5}$$

Now, observe that one can always pick $N_n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{1}{N_n+1}\left(\frac{R}{|a_n|}\right)^{N_n+1}$ converges: $N_n = n$ does the job. The Eq.(5) tells us that $\sum_{n=1}^{\infty} \psi_n(z)$ is absolutely and uniformly convergent on $\overline{D}_R(0)$, so that $\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{p_n(z)}$ is an analytic function on $D_R(0)$ for all $R > 0$ $\square$

Corollary: Every function which is meromorphic on all of $\mathbb{C}$ is the quotient of two entire functions.

Indeed, if f is meromorphic on $\mathbb{C}$, the theorem enables us to construct an entire function g whose zeros are the poles of f. $F(z) := f(z)g(z)$ is then an entire function, and

$$f(z) = \frac{F(z)}{g(z)}$$

3.4 Genus of a canonical product

From the proof of the Weierstrass factorization theorem, we know that

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h} \tag{6}$$

(observe that in the expression, h is independent of n!) represents an entire function if

$$\frac{1}{h+1}\sum_{n=1}^{\infty}\left(\frac{R}{|a_n|}\right)^{h+1}$$

converges for all R, which is the case if $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{h+1}}$ converges.
Consider the smallest integer h for which this is the case. For this h, (6) is called the *canonical product associated with the sequence* $(a_n)_{n\in\mathbb{N}}$, and h is called the *genus* of the canonical product.

Imagine f has a Weierstrass factorization for which the infinite product is a canonical product. If in this representation, g as defined in the factorization theorem reduces to a polynomial, f is said to be of *finite genus*. The genus of f is then defined to be the max between the genus of the canonical product and the degree of the polynomial.

An equivalent, slightly more general definition of the genus of an entire function f with zeros a_n is as follows: the genus h of f is the smallest integer such that f can be represented in the form

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$

where g is a polynomial of degree less than or equal to h.
If there is no such representation for f, the genus of f is infinite.

Examples
• Any entire function of genus zero is of the form

$$f(z) = Az^M \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) \quad A \in \mathbb{C}$$

• The canonical representation of genus 1 entire functions can have either of the following two forms:

1. $$f(z) = Bz^M e^{\alpha z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \quad B \in \mathbb{C}^* \ , \ \alpha \in \mathbb{C}$$

with $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ nonconvergent and $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ convergent.

1. $$f(z) = Cz^M e^{\alpha z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \quad C \in \mathbb{C}^* \ , \ \alpha \in \mathbb{C}^*$$

and $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ convergent.

3.5 Order of an entire function

Definition: Let f be an entire function, and $M(R)$ the maximum of $|f(z)|$ on $|z| = R$. The order λ of f is defined by

$$\lambda = \limsup_{R\to\infty} \frac{\ln \ln M(R)}{\ln R} \tag{7}$$

In other words, λ is the smallest number such that

$$M(R) \leq e^{R^{\lambda+\epsilon}}$$

for all $\epsilon > 0$, as soon as R is large enough.

It can be shown that the genus and the order of an entire function are closely related. This is the point of Hadamard's factorization theorem.

Theorem (Hadamard Factorization Theorem): The genus h and the order λ of an entire function satisfy the double inequality $h \leq \lambda \leq h + 1$.

The proof of this theorem is somewhat lengthy, and we will skip it in this course. You can however find the key steps of the proof in Ahlfors' textbook.

Corollary: An entire function of fractional order assumes every finite value infinitely many times.

Proof: $\forall z_0 \in \mathbb{C}$, f and $f - z_0$ have the same order. Thus, to prove the corollary we just want to show than an entire function with fractional order has infinitely many zeros.
Let us assume that f has a finite number of zeros. Then there exists a polynomial p such that $F = f/p$ does not have any zeros, and has the same order λ as f. Hadamard's factorization theorem then tells us that $F = e^{g(z)}$, where g is a polynomial of degree h such that $h \leq \lambda \leq h + 1$. Now, it is clear that the order of $F = e^{g(z)}$ is h itself, which is an integer. We have a contradiction, which proves the corollary □