1 Conformality

1.1 Preservation of angle

The open mapping theorem tells us that an analytic function such that $f'(z_0) \neq 0$ maps a small neighborhood of z_0 onto a neighborhood of $f(z_0)$ in a one-to-one fashion. In particular, f maps continuously differentiable arcs through z_0 onto continuously differentiable arcs through $f(z_0)$. We now show that f preserves angles between two such arcs.

Suppose f is a complex function (not necessarily analytic) defined on a neighborhood of z_0 and such that $f(z) \neq f(z_0)$ for all $z \neq z_0$ in that neighborhood. If there exists $w = e^{i\varphi} \in \mathbb{C}$, with $\varphi \in \mathbb{R}$ such that for all $\theta \in \mathbb{R}$ and $r \in \mathbb{R}_+^*$,

$$\frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} \xrightarrow[r \to 0^+]{} e^{i\varphi} e^{i\theta}$$

then we say that f *preserves angles at* z_0.

Theorem: Suppose f is analytic at z_0. Then f preserves angles at z_0 iff $f'(z_0) \neq 0$.

Proof: Let f be analytic in a neighborhood of z_0, and such that $f'(z_0) \neq 0$.

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \mathbb{R}\ ,\ \lim_{r \to 0^+} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} = e^{i\theta} \lim_{r \to 0^+} \frac{\frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}}}{\frac{|f(z_0 + re^{i\theta}) - f(z_0)|}{r}} = e^{i\theta} \frac{f'(z_0)}{|f'(z_0)|}$$

So we see that if w exists, $w = f'(z_0)/|f'(z_0)|$.

Conversely, suppose that $f'(z_0) = 0$. If f is constant, f does not preserve angle. If f is not constant, there exists $N \in \mathbb{N}^*$ such that $f(z) = f(z_0) + (z - z_0)^N g(z)$, with g analytic at z_0, and $g(z_0) \neq 0$. In that case,

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \mathbb{R}\ ,\ \frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} = e^{iN\theta} \frac{g(z_0 + re^{i\theta})}{|g(z_0 + re^{i\theta})|} \xrightarrow[r \to 0^+]{} e^{i\theta} e^{i(N-1)\theta} \frac{g(z_0)}{|g(z_0)|}$$

We see that f increases angle by a factor of N, so f does not preserve angles $\square$

Definition: A function which is analytic on an open connected set Ω and has a nonvanishing derivative is called a *conformal map*.

Examples: • exp maps any arbitrary vertical line $\{z : \Re(z) = x_0 \in \mathbb{R}\}$ onto the circle with center 0 and radius e^{x_0}. exp maps any horizontal line $\{z : \Im(s) = y_0 \in \mathbb{R}$ onto the open ray from 0 through e^{iy_0}. We see that exp preserves the orthogonality of these curves.

• The function $f(z) = z^2$ maps two curves crossing at 0 with angle α into two curves crossing at 0 with angle 2α. It is not conformal at $z = 0$.

Note: The definition of angle preservation contains the preservation of the magnitude of the angle *as well as* its orientation.
You can easily convince yourself that the mapping by the complex conjugate of an analytic function whose derivative does not vanish preserves the magnitude of the angle, but reverses the orientation. It is called an *indirectly conformal map*.

1.2 Length and area

• Consider an analytic function f such that $f'(z_0) \neq 0$.

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

Any small line segment with one end point at z_0 is expanded by an amount $|f'(z_0)|$. This expansion is independent of the direction of the line segment.

- Consider a set Ω in $\mathbb{R}^2$. Its area is given by

$$A(\Omega) = \iint_\Omega dxdy$$

The mapped set $\Omega' = f(\Omega)$, where $f = (u(x,y), v(x,y))$ is a bijective differentiable mapping, has an area given by

$$A(\Omega') = \iint_{\Omega'} dudv = \iint_\Omega |u_x v_y - u_y v_x| dxdy$$

Now, if $f(z) = u(x,y) + iv(x,y)$ is a conformal mapping on an open set containing Ω, f satisfies the Cauchy-Riemann relations, so that

$$|u_x v_y - u_y v_x| = |f'(z)|^2$$

and therefore

$$A(\Omega') = \iint_\Omega |f'(z)|^2 dxdy$$

Infinitesimal areas are expanded by the factor $|f'(z)|^2$.

2 Linear fractional transformations

2.1 Möbius transformation

As we have already seen in Lecture III, *Möbius transformations*, also called *linear fractional transformations*, are maps of the form

$$S(z) = \frac{az+b}{cz+d}$$

with $(a,b,c,d) \in \mathbb{C}^4$ such that $ad - bc \neq 0$ in order to avoid the situation in which S is a constant function. The domain of S is $\mathbb{C} \setminus \{-\frac{d}{c}\}$ if $c \neq 0$, and $\mathbb{C}$ if $c = 0$.

One often defines S on the extended complex plane $\hat{\mathbb{C}}$ by setting

$$S\left(-\frac{d}{c}\right) = \infty \quad , \quad S(\infty) = \frac{a}{c} \text{ if } c \neq 0 \quad , \quad S(\infty) = \infty \text{ if } c = 0$$

For all $z \in \mathbb{C} \setminus \{-\frac{d}{c}\}$,

$$S'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

so S is a conformal map on its domain.

Lastly, we have already seen that $\forall\ (a,b,c,d) \in \mathbb{C}^4$ such that $ad - bc \neq 0$, S has an inverse:

$$z = S^{-1}(w) = \frac{dw-b}{-cw+a}$$

2.2 The linear group

Theorem: The set $\mathcal{M}$ of Möbius transformations is a group under composition. Any Möbius transformation is a composition of the following maps:

(1) Translation: $z \mapsto z + a$, with $a \in \mathbb{C}$ constant

(2) Scaling and rotation: $z \mapsto kz$, $k \in \mathbb{C}^*$ constant

(3) Inversion: $z \mapsto \frac{1}{z}$

• Before we discuss the group structure, let us prove the last point of the theorem. If $c \neq 0$

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+\frac{d}{c})} + \frac{a}{c}$$

which is the composition of a translation, an inversion, a scaling and rotation, and another translation. If $c = 0$, $S(z) = \frac{a}{d}z + \frac{b}{d}$ is the composition of a scaling and rotation and a translation.

• Regarding the group structure of $\mathcal{M}$, we have already done most of the work:

- $S(z) = z$ is the identity
- Any S has an inverse S^{-1}
- If $S_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \in \mathcal{M}$ and $S_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2} \in \mathcal{M}$, then

$$S_1(S_2(z)) = \frac{Az+B}{Cz+D}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

which demonstrates that the composition is also associative.

2.3 The cross ratio

Let z_2, z_3, and z_4 be distinct points in $\hat{\mathbb{C}}$. To these points, we associate the Möbius transformation

$$S(z) = \begin{cases} \frac{\frac{z-z_3}{z-z_4}}{\frac{z_2-z_3}{z_2-z_4}} & \text{if } (z_2, z_3, z_4) \in \mathbb{C}^3 \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty \end{cases}$$

Observe that $\forall (z_2, z_3, z_4) \in \hat{\mathbb{C}}^3$, $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$. We now show that S is the unique such linear fractional transformation, with the following general result.
For any two sets of distinct complex numbers $\{z_2, z_3, z_4\}$ and $\{w_2, w_3, w_4\}$ in the extended complex plane, there exists a unique Möbius transform taking z_n to w_n, for $n \in \{2, 3, 4\}$. To prove this, we use the lemma below.

Lemma: A Möbius transform can have at most two fixed points, unless S is the identity map.

Proof of the lemma: A fixed point z_0 of S is such that

$$\frac{az_0+b}{cz_0+d} = z_0 \quad \Leftrightarrow \quad cz_0^2 + (d-a)z_0 - b = 0$$

This polynomial equation has at most two solutions, which concludes our proof.

Proof of the existence and uniqueness of the Möbius transformation: Let us call $S_{z_2 z_3 z_4}$ the Möbius transform defined on the previous page. $S_{z_2 z_3 z_4}$ maps $\{z_2, z_3, z_4\}$ to $\{1, 0, \infty\}$. The inverse map of S_{w_2, w_3, w_4} takes $\{1, 0, \infty\}$ to $\{w_2, w_3, w_4\}$. Hence, $S^{-1}_{w_2, w_3, w_4} \circ S_{z_2 z_3 z_4}$ takes $\{z_2, z_3, z_4\}$ to $\{w_2, w_3, w_4\}$.
Now, let us imagine that there are two linear fractional transformations S and T sending $\{z_2, z_3, z_4\}$ to $\{w_2, w_3, w_4\}$. Then, $\forall n \in \{2, 3, 4\}$, $S \circ T^{-1}(w_n) = S(z_n) = w_n$. $S \circ T^{-1}$ has three fixed points, so $S = T$ $\square$

<u>**Definition**</u>: The cross ratio (z_1, z_2, z_3, z_4) is the image of $z_1 \in \hat{\mathbb{C}}$ under the unique Möbius transformation which maps $\{z_2, z_3, z_4\}$ to $\{1, 0, \infty\}$

2.4 Circlelines

Proposition: Let r and s be real numbers, and $k \in \mathbb{C}$. The equation $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$

- represents a line if $r = 0$ and $k \neq 0$
- represents a circle if $r \neq 0$ and $|k|^2 \geq rs$, with equation

$$\left|z + \frac{k}{r}\right| = \frac{1}{r}\sqrt{|k|^2 - rs}$$

This result can be easily proved by writing $z = x + iy$ and expanding in x and y.

Definition: The locus of the points of $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$, if non-empty, is called a *circleline*.

Note: The definition above may feel a bit odd at first, in the sense that it tries to combine under the same term two different objects: lines and cirles.
The reason why the definition makes sense in our context is that both circles and extended lines in the complex plane correspond to circles on the Riemann sphere, as discussed in Lecture I. Some authors, including Ahlfors, do not even use the term circleline, and call the locus of the points above a circle.

Lemma: Let $r \in \mathbb{R}$, and $(z_1, z_2) \in \mathbb{C}^2$, with $z_1 \neq z_2$. The locus of the equation $|z - z_1| = r|z - z_2|$ represents a circle if $r \neq 1$, and a line if $r = 1$, namely the line that is perpendicular to the line segment $[z_1, z_2]$ and passes through its midpoint.

Theorem: A Möbius transformation maps a circleline to a circleline.

Proof: Since any Möbius transformation is the composition of a translation, a scaling and rotation, and an inversion, we just have to show that the theorem holds independently for each of these transformations.

• The image of $r|z|^2 + \overline{k}z + k\overline{z} + s = 0$ under the translation $z \mapsto w = z + a$ is

$$r|w-a|^2 + \overline{k}(w-a) + k\overline{w-a} + s = 0 \quad \Leftrightarrow \quad r|w|^2 + (\overline{k-a})w + (k-a)\overline{w} + r|a|^2 - (\overline{k}a + k\overline{a}) + s = 0$$

The last three terms are real numbers, so this is indeed the equation of a circleline.

• The result is immediate for a scaling and rotation, which corresponds to multiplication by a nonzero complex number

• For the inversion, we distinguish the case in which the circleline is a circle, and the case in which the circleline is a line

(i) Say the circleline is a circle, with equation $|z - a| = r$. If $a \neq 0$, the image of this circle under inversion is

$$\left|\frac{1}{w} - a\right| = r \quad \Leftrightarrow \quad \left|w - \frac{1}{a}\right| = \frac{r}{|a|}|w|$$

and from the previous lemma we know that the equation on the right-hand side is the equation of a circleline.

If $a = 0$, the image of $|z| = r$ under inversion is the circle $|w| = 1/r$.

(ii) Let us now consider the line $\overline{k}z + k\overline{z} + s = 0$. Under inversion, this becomes

$$\frac{\overline{k}}{w} + \frac{k}{\overline{w}} + s = 0 \quad \Leftrightarrow \quad s|w|^2 + \overline{k}\overline{w} + kw = 0$$

which is a circleline □

Theorem: Let z_1, z_2, z_3, z_4 be distinct points in $\hat{\mathbb{C}}$. The cross ratio (z_1, z_2, z_3, z_4) is a real number iff the four points lie in a circleline.

Proof: If $(z_1, z_2, z_3, z_4) \in \mathbb{R}$, $S_{z_2 z_3 z_4}$ maps $\{z_1, z_2, z_3, z_4\}$ to points on the extended x-axis. $S^{-1}_{z_2 z_3 z_4}$ sends $\{(z_1, z_2, z_3, z_4), 1, 0, \infty\}$ to $\{z_1, z_2, z_3, z_4\}$. A Möbius transformation takes the extended line to a circleline, so $\{z_1, z_2, z_3, z_4\}$ lie in a circleline.

Conversely, if the four points lie in a circleline, $S_{z_2 z_3 z_4}$ sends the circleline to a circleline, which is the x-axis. So $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ $\square$.