The point of this lecture is to prove that the unit disk can be mapped conformally onto any simply connected open set in the plane, other than the plane itself. We will then know that any two such simply connected open set can be mapped conformally onto each other, since we can map either of these sets to the unit disk.

The result above is known as the Riemann mapping theorem. We will prove it using basic theory of normal families. We start this lecture with that.

1 Normal families

1.1 Definition

Let Ω be an open connected set in $\mathbb{C}$ and $\mathcal{F}$ a family of analytic functions on Ω. $\mathcal{F}$ is said to be

(i) normal or relatively compact if every sequence from $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω

(ii) locally uniformly bounded if for any compact subset K of Ω, there exists $M > 0$ such that $|f(z)| \leq M$ $\forall f \in \mathcal{F}$, $\forall z \in K$.

(iii) equicontinuous on a compact set K of Ω if $\forall\ \epsilon > 0$, $\exists \delta > 0$ such that

$$\forall (z, w) \in K^2\ , \qquad |z - w| < \delta \ \Rightarrow\ |f(z) - f(w)| < \epsilon \quad \forall f \in \mathcal{F}$$

Observe that if there exists $M > 0$ such that $\forall\ z \in \Omega$, $\forall\ f \in \mathcal{F}$, $|f'(z)| \leq M$, then $\mathcal{F}$ is equicontinuous.

1.2 Equivalence of the definitions

Theorem (Montel's theorem): Suppose $\mathcal{F}$ is a locally uniformly bounded family of analytic functions in an open connected set Ω. Then $\mathcal{F}$ is normal.

Proof: The proof of Montel's theorem can be decomposed into the following two steps:

(1) **Lemma**: Suppose $\mathcal{F}$ is a family of analytic functions uniformly bounded on compact subsets of Ω. Then $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

To show this, let us consider a compact subset K of Ω. There exists $r > 0$ such that the set $\tilde{K}$ defined by

$$\tilde{K} := \{z \in \mathbb{C} : d(z, K) \leq 2r\}$$

is contained in Ω. Let

$$M = \sup_{f \in \mathcal{F}, z \in \tilde{K}} |f(z)|$$

and $(z, w) \in K^2$ such that $|z - w| < r$.

The closed disk $\overline{D}_{2r}(z)$ is contained in Ω. If we call γ the boundary of this disk, we may write

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)(z - w)}{(\zeta - w)(\zeta - z)} d\zeta$$

$\forall \epsilon > 0$, let $\delta = \min\left(\epsilon r/M, r\right)$.

$$|z - w| < \delta \ \Rightarrow\ |f(z) - f(w)| \leq \frac{1}{2\pi} \frac{4\pi r}{2r^2} M |z - w| \leq \epsilon$$

Hence $\mathcal{F}$ is equicontinuous on K.

(2) Now that we know that $\mathcal{F}$ in Montel's theorem is equicontinuous on every compact subset of Ω, we can prove the theorem.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence from $\mathcal{F}$. Consider a sequence $(w_j)_{j\in\mathbb{N}}$ which is everywhere dense in Ω. The sequence $(f_n(w_1))_{n\in\mathbb{N}}$ is bounded, so we can extract a convergent subsequence, and write the corresponding functions $(f_{n,1})_{n\in\mathbb{N}}$. $(f_{n,1}(w_2))_{n\in\mathbb{N}}$ is bounded, so we can extract another convergent subsequence $(f_{n,2})_{n\in\mathbb{N}}$. Repeating the process, we obtain the nested sequence $(f_{n,k})_{k\in\mathbb{N}^*,n\in\mathbb{N}}$ such that $(f_{n,k}(w_k))_{n\in\mathbb{N}}$ converges for every w_j with $j \in [\![1,k]\!]$.

Let $g_n = f_{n,n}$ be the diagonal sequence of functions. g_n converges at all the points $(w_n)_{n\in\mathbb{N}}$. We now show that $(g_n)_n \in \mathbb{N}$ converges uniformly on every compact subset K of Ω.

Let K be such a compact subset. Since $\mathcal{F}$ is equicontinuous on K, $\forall\epsilon > 0\ \exists\delta > 0$ such that

$$\forall(z,w) \in K^2\ ,\ |z-w| < \delta\ \Rightarrow\ |f(z) - f(w)| < \frac{\epsilon}{3}$$

Since K is compact, we can select N points from $(w_j)_{j\in\mathbb{N}}$ such that $K \subset \cup_{p=1}^{N} D_\delta(w_p)$. By construction of the functions g_n, for that choice of ϵ, there exists $N_\epsilon \in \mathbb{N}$ such that

$$m,n \geq N_\epsilon \quad\Rightarrow\quad \forall p \in [\![1,N]\!], |g_n(w_p) - g_m(w_p)| < \frac{\epsilon}{3}$$

Now, $\forall w \in K$, $\exists p_0$ such that $w \in D_\delta(w_{p_0})$. Then, if $m,n \geq N_\epsilon$,

$$|g_n(w) - g_m(w)| \leq |g_n(w) - g_n(w_{p_0})| + |g_n(w_{p_0}) - g_m(w_{p_0})| + |g_m(w_{p_0}) - g_m(w)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

from which we conclude that g_n converges uniformly on K, as it is uniformly Cauchy on K.

2 The Riemann mapping theorem

2.1 Statement of the theorem

Theorem (Riemann mapping theorem): Given any simply connected open set Ω which is not the whole plane, and a point z_0 in Ω, there exists a unique analytic function f in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that f defines a one-to-one mapping of Ω onto the disk $|w| < 1$.

2.2 Proving the theorem

Uniqueness: Suppose there are two such functions f_1 and f_2. Then $f_1 \circ f_2^{-1}$ is a one-to-one mapping of the unit disk to itself. You will prove in Homework 9, using the lemma of Schwarz, that this implies that $f_1 \circ f_2^{-1}$ is a Möbius transformation of the form

$$S(z) = \lambda\frac{z-a}{1-\bar{a}z}$$

with $|\lambda| = 1$ and $a \in \mathbb{C}$ such that $|a| < 1$. Then the normalization conditions $S(0) = 0$, $S'(0) > 0$ imply that $S(z) = z$, so that $f_1 \equiv f_2$.

Existence:

- **Lemma**: If Ω is a simply connected open set which is not $\mathbb{C}$, there exists a one-to-one conformal map $h : \Omega \mapsto h(\Omega)$ such that $h(\Omega)$ does not intersect a disk $D_\delta(w_0)$ for some $w_0 \in \mathbb{C}$ and $\delta > 0$.

  To prove the lemma, observe that by hypothesis, there exists $a \in \mathbb{C} \setminus \Omega$. Hence $\varphi(z) = z - a$ is a nonvanishing analytic function on Ω. From Lecture 9, we then know that we can define an analytic function h on Ω such that $h^2(z) := z - a$.

  Observe that h does not take the same value twice, nor opposite values. Indeed if $(z_1, z_2) \in \Omega^2$ are such that $h(z_1) = \pm h(z_2)$, then $z_1 - a = z_2 - a\ \Rightarrow z_1 = z_2$.

  Now, by the open mapping theorem, $\forall z_0 \in \Omega$, $h(\Omega)$ contains a disk $D_\delta(h(z_0))$. This means that $h(\Omega)$ does not contain the disk $D_\delta(-h(z_0))$, which proves the lemma, with $w_0 = -h(z_0)$.

- We now show that there are one-to-one, analytic functions g from Ω to $D_1(0)$ such that $g(z_0) = 0$ and $g'(z_0) > 0$.

To see that there is at least one such function, consider the function G_0 on Ω defined by

$$G_0(z) = \frac{\delta}{4}\frac{|h'(z_0)|}{|h(z_0)|^2}\frac{h(z_0)}{h'(z_0)}\frac{h(z)-h(z_0)}{h(z)+h(z_0)}$$

with δ and h as defined in the previous lemma. G_0 is one-to-one since h is and G_0 is obtained from h by a linear fractional transformation.

$$G_0(z_0) = 0 \ \ , \ \ G_0'(z_0) = \frac{\delta}{8}\frac{|h'(z_0)|}{|h(z_0)|^2} > 0$$

Finally, observe that

$$\forall z \in \Omega \ , \ |G_0(z)| = \frac{\delta}{4}\frac{1}{|h(z_0)|}\left|\frac{h(z)-h(z_0)}{h(z)+h(z_0)}\right| = \frac{\delta}{4}\left|\frac{1}{h(z_0)} - \frac{2}{h(z)+h(z_0)}\right| < 1$$

This proves our point.

- The last step of the proof is to show that within the family $\mathcal{F}$ of functions g discussed previously, there exists an f with maximal derivative, and that this f has all the properties required for the mapping theorem. The key is to show that f is surjective.

Observe that by Cauchy's estimate on $D_r(z_0) \subset \Omega$, $\forall\, g \in \mathcal{F}$,

$$|g'(z_0)| \leq \frac{1}{r}$$

The set $\{|g'(z_0)| \ , \ g \in \mathcal{F}\}$ is therefore bounded, and has a supremum $B = \sup_{g\in\mathcal{F}} |g'(z_0)|$. There exists a sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that $g_n'(z_0) \to B$ as $n \to +\infty$.

Now, since $|g_n| < 1$ on Ω, by Montel's theorem we know that there exists a subsequence (g_{n_k}) of g_n converging to a function f uniformly on every compact subset of Ω. By Weierstrass' theorem, f is analytic. Furthermore,

$$f(z_0) = \lim_{n_k\to\infty} g_{n_k}(z_0) = 0 \quad , \quad |f'(z_0)| = \lim_{n_k\to\infty} |g_{n_k}'(z_0)| = B > 0$$

From the latter result, we can say that f is not a constant function.

Now, we take some z_1 in Ω, and define $\tilde{g}_{n_k} := g_{n_k}(z) - g_{n_k}(z_1)$. $\tilde{g}_{n_k}$ is nonzero on $\Omega \setminus \{z_1\}$. Hence, by Hurwitz' theorem, $\tilde{f} := f(z) - f(z_1)$ is nonzero on $\Omega \setminus \{z_1\}$. We have just proved that f is one-to-one on Ω.

The remaining question is: is f onto?

Suppose it is not: $\exists w_0 \in D_1(0)$ such that $w_0 \notin f(\Omega)$. Then, as before, we can consider the single-valued function F on Ω such that

$$F^2(z) := \frac{f(z) - w_0}{1 - \overline{w_0}f(z)} \tag{1}$$

F is one-to-one and satisfies $|F(z)| < 1$ for $z \in \Omega$. F can be normalized as follows:

$$G(z) := \frac{|F'(z_0)|}{F'(z_0)}\frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$$

G is one-to-one, satisfies $|G(z)| < 1 \ \forall z \in \Omega$, and $G(z_0) = 0$. Furthermore,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{|f'(z_0)|}{2\sqrt{|w_0|}}(1 - |w_0|^2)\frac{1}{1 - |w_0|} = |f'(z_0)|\frac{1 + |w_0|}{2\sqrt{|w_0|}} > |f'(z_0)|$$

This is a contradiction, so we may say that f is onto, which concludes the proof of the theorem □

2.3 Returning to our definition of simple connectedness

You may recall that in Lecture 9, we introduced an unusual definition of simple connectedness, only valid in $\mathbb{R}^2$, but very convenient for our purposes at the time. I told you then that we would eventually be able to prove the equivalence of the more conventional definition – that any simple closed curve can be shrunk to a point continuously in the set. We are now ready for this.

We have already shown in Lecture 9 that if every closed curve in Ω is contractible to a point, then Ω is simply connected in the sense of Ahlfors. It remains to prove that if Ω is simply connected in the sense of Ahlfors, every closed curve in Ω is contractible to a point.
This is easy to see: based on the definition of simple connectedness of Ahlfors, we were able to construct the square root h of φ and the function F in (1), and thus prove the Riemann mapping theorem. Therefore, if Ω is simply connected in the sense of Ahlfors, it is homeomorphic to the unit disk $D_1(0)$, which is simply connected in the usual sense $\square$

3 Visualizing maps

A streamlined way to visualize the transformation of a set by a map is to consider the effect of the map on a mesh of the set. This is what we do below.

• $f(z) = z^2$
We know that f is conformal on $\mathbb{C} \setminus \{0\}$. We consider the mesh

$$\begin{cases} z_{x_0} = x_0 + iy \\ z_{y_0} = x + iy_0 \end{cases}$$

for a countable set of x_0 and y_0. Any vertical line $z_{x_0} = x_0 + iy$ is mapped to

$$w = z_{x_0}^2 = u_{x_0}^2 + iv_{x_0}^2 = x_0^2 - y^2 + 2ix_0 y$$

We see that the real and imaginary parts satisfy

$$v_{x_0}^2 = 4x_0^2 y^2 = 4x_0^2(x_0^2 - u)$$

This is the equation of a parabola in the w-plane, with focus $(0, 0)$ and pointed in the negative direction.

Any horizontal line $z_{y_0} = x + iy_0$ is mapped to $w = x^2 - y_0^2 + 2ixy_0$. Hence

$$v_{y_0}^2 = 4y_0^2(u + y_0^2)$$

This is the equation of a parabola in the w-plane, with focus $(0, 0)$ and pointed in the positive direction.
A mesh in the z-plane and its maps in the w-plane by the function $f(z) = z^2$ is shown in Figure 1.

• *The Cayley transform*
Consider the Möbius transformation

$$S(z) = \frac{z - i}{z + i}$$

sometimes called the Cayley transform. By direct computation, it is easy to see that S maps the real line to the unit circle. Another way to see this is that the points $\{\infty, 1, -1\}$ are mapped to $\{1, -i, i\}$. Since S maps circlelines to circlelines, we have the desired result.
Using the same theorem, it is easy to see that every horizontal line $z = x + iy_0$, with $y_0 > 0$ is mapped to a circle inside the unit circle. By direct computation, one finds that this is a circle of center $(y_0/(y_0 + 1), 0)$ and radius $1/(y_0 + 1)$.
Likewise, every vertical line $z = x_0 + iy$ is mapped to a circle of center $(1, -1/x_0)$ and radius $1/x_0$. The arcs of these circles corresponding to $y > 0$ are inside the unit disk.
The Cayley transform maps the upper half plane onto the unit disk.
A few mesh lines in the upper half of the z-plane and their maps in the w-plane by the Cayley transform are shown in Figure 2.

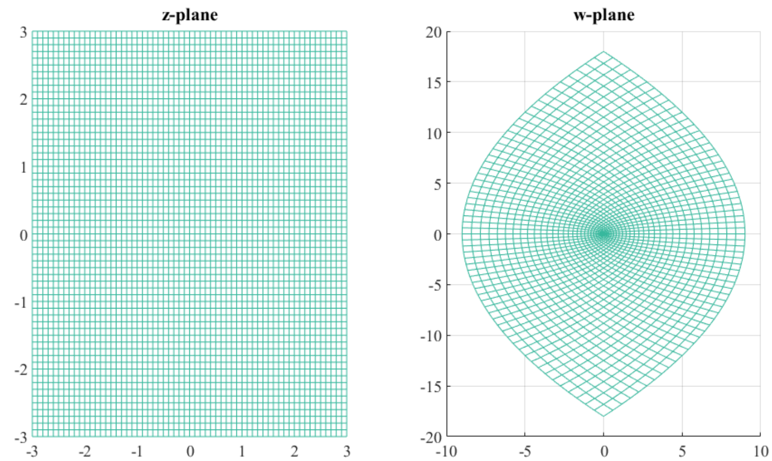

Figure 1: Mesh in the z-plane and its image by the map $z \mapsto z^2$

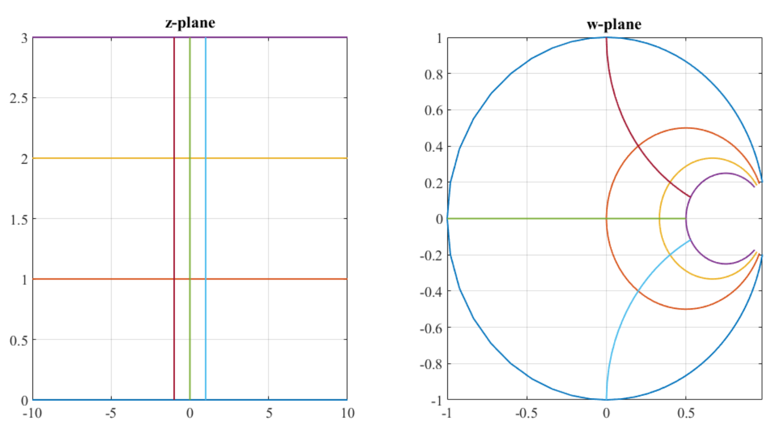

Figure 2: Mesh lines in the upper half of the z-plane and their images by the Cayley transform

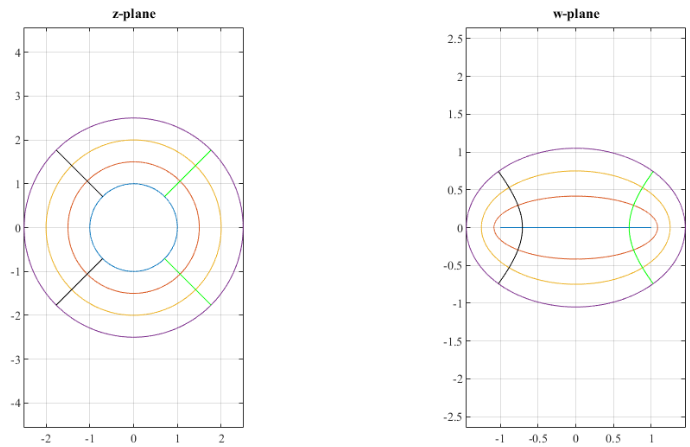

Figure 3: (Left) Circles $z = r_0 e^{i\theta}$, $\theta \in [0, 2\pi]$ in the z-plane, with $r_0 \in \{1, 1.5, 2, 2.5\}$ and rays $x = \pm y$, and (Right) their images in the w-plane under the Joukowski map $J(z) = 1/2(z + 1/z)$

- *The Joukowski map*

A mapping which was historically important in fluid dynamics is the Joukowski map, defined by

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

This map is conformal everywhere on $\mathbb{C}$ except at $z = 0$ where is it not defined, and at $z = \pm 1$, where $dw/dz = 0$.

Observe that since $f(z) = f(1/z)$, two points $(z_1, z_2) \in \mathbb{C}^2$ such that $z_1 z_2 = 1$ are mapped onto the same image by f. f is two-to-one, and only one-to-one in a domain Ω if there are no two points z_1 and z_2 such that $(z_1, z_2) \in \mathbb{C}^2$ and $z_1 z_2 = 1$. f is for example one-to-one in $D_1(0)$, and in the exterior of $D_1(0)$.

To further visualize the map, let us consider families of circles $z = r_0 e^{i\theta}$, with $r_0 > 0$.

$$w = u + iv = \frac{1}{2}\left(r_0 e^{i\theta} + \frac{1}{r_0} e^{-i\theta}\right) \qquad \Rightarrow \qquad u = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos\theta \quad , \quad v = \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin\theta$$

Hence, the circle $C_{r_0}(0)$, $r_0 < 1$ is mapped onto an ellipse, and this ellipse degenerates into the line segment $[-1, 1]$ as $r_0 \to 1$.
Observe that $D_1(0)$ is mapped to $\mathbb{C} \setminus \{[-1, 1]\}$, and since $f(z) = f(1/z)$, the exterior of $D_1(0)$ is also mapped to $\mathbb{C} \setminus \{[-1, 1]\}$. The images of circles with radius $r_0 \geq 1$ and the images of the rays $x = \pm y$ perpendicular to the circles under the Joukowski map are shown in Figure 3 to illustrate this.
This mapping has relevance in fluid dynamics because the image of certain circles not centered at the origin under the map resembles the cross section of an airplane wing. You can for instance see the image of the circle $z = 0.1 + 0.2i + 0.85e^{i\theta}$, $\theta \in [0, 2\pi]$ under the Joukowski map in Figure 4.
The idea, then, at a time when numerical solvers for fluid dynamics were not available or very slow, was to compute the tractable problem of fluid potential flow around a cylinder with circular cross section and apply the Joukowski mapping to the solution to obtain the fluid flow around the airplane wing. We will look at this in the next lecture.

4 Conformal mapping of an annulus

The Riemann mapping theorem applies to simply connected open sets. In the proof of the theorem, that was the condition under which we could construct the analytic function h. Topologically, it is intuitive that the theorem cannot apply to non-simply connected domains: a hole cannot be made to disappear under a one-to-one continuous mapping.

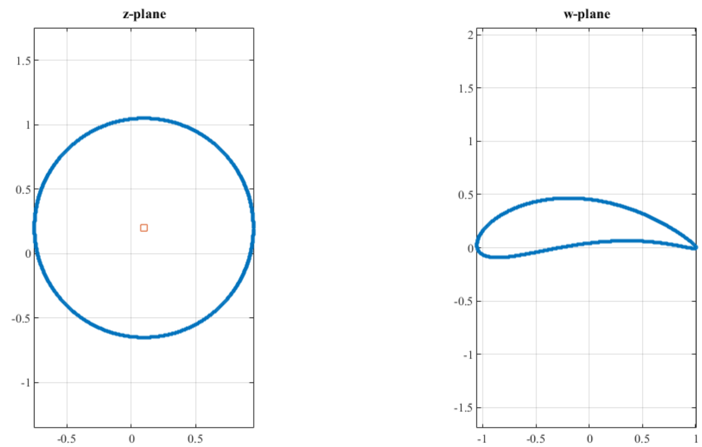

Figure 4: (Left) Circle $z = 0.1 + 0.2i + 0.85e^{i\theta}$, $\theta \in [0, 2\pi]$ in the z-plane (its center is shown with the red square marker), and (Right) its image in the w-plane under the Joukowski map $J(z) = 1/2(z + 1/z)$

One may however ask if one can generalize the Riemann mapping theorem to non-simply connected domains by looking into mappings of non-simply connected open sets to the simplest non-simply connected set one can think of, i.e. the annulus. Even more simply, one can ask whether any two concentric annuli are conformally equivalent. The answer to this question is, perhaps surprisingly, negative.

Theorem: $A(r_1, R_1) := \{z : r_1 < |z| < R_1\}$ and $A(r_2, R_2) := \{z : r_2 < |z| < R_2\}$ are conformally equivalent iff $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.

Proof: Without loss of generality, let us rescale $A(r_1, R_1) = A_1$ and $A(r_2, R_2) = A_2$ such that $A_1 = A(1, R_1)$ and $A_2 = A(1, R_2)$. Let us assume there is an analytic conformal map f such that $f(A_1) = A_2$. Then, since f is a homeomorphism between A_1 and A_2, either $|f(z)| \to 1$ as $|z| \to 1$ or $|f(z)| \to R_2$ as $|z| \to 1$. If the second situation holds, we can always define $g := R_2/f$ which is such that $|g(z)| \to 1$ as $|z| \to 1$. Hence, without loss of generality, we assume that $\lim_{|z|\to 1} |f(z)| = 1$ and thus $\lim_{|z|\to R_1} |f(z)| = R_2$. Now, since $f(z) \neq 0$ $\forall z \in A_1$, $\ln |f|$ is harmonic in A_1. Let

$$u(z) := \ln |f(z)| - \alpha \ln |z| \ , \ \alpha = \frac{\ln R_2}{\ln R_1} \quad , \quad \forall z \in A_1$$

Observe that

$$\lim_{|z|\to 1} u(z) = 0 = \lim_{|z|\to R_1} u(z)$$

We can extend u to a continuous function on $\overline{A}_1$, with $u = 0$ on ∂A_1. Since u is harmonic in A_1, $u \equiv 0$ on A_1. Therefore,

$$\forall z \in A_1 \ , \ |f(z)| = |z|^\alpha \tag{2}$$

Now, take $z_0 \in A_1$, and $r > 0$ such that $D_r(z_0) \subset A_1$. As we have shown in Homework 1 (or by the Maximum Modulus Principle), Eq.(2) implies that

$$\forall z \in D_r(z_0) \ , \ f(z) = e^{i\theta_0} z^\alpha \tag{3}$$

for some $\theta_0 \in \mathbb{R}$.
Taking the logarithmic derivative of Eq.(3), we find

$$\forall z \in D_r(z_0) \quad , \quad \frac{f'(z)}{f(z)} = \frac{\alpha}{z}$$

And since this is true for any $z_0 \in A_1$ (provided r is chosen small enough that $D_r(z_0) \subset A_1$)

$$\forall z \in A_1 \quad , \quad \frac{1}{2\pi i} \frac{f'(z)}{f(z)} = \frac{1}{2\pi i} \frac{\alpha}{z}$$

We integrate this equality over the circle of center 0 and radius $R_1^{1/2}$. The right-hand side gives α. The left-hand side is

$$\frac{1}{2\pi i}\int_{f(\gamma)}\frac{dw}{w}=\pm 1$$

and so it must be that $\alpha = 1$, i.e. $R_1 = R_2$.

Conversely, consider $A_1(r_1, R_1)$ and $A_2(r_2, R_2)$ such that $\frac{R_1}{r_1} = \frac{R_2}{r_2} = \beta$. Let $\gamma = r_2/r_1$. The mapping $z \mapsto \gamma z$ transforms $A_1(r_1, R_1) = A_1(r_1, \beta r_1)$ into $A(\gamma r_1, \gamma\beta r_1) = A_2(r_2, R_2)$.
This concludes our proof.

Note finally that it can be shown that any doubly connected region of $\mathbb{C}$ can be conformally mapped to an annulus. The proof of this result is however beyond the scope of this class.
From what we have just seen, the annulus to which the doubly connected region is mapped has a ratio r/R which is uniquely specified.

5 Conformal mapping of the unit disk to polygons

5.1 Mapping the unit disk to a polygon

The Riemann mapping theorem tells us that one can map any polygon to the unit disk. What is interesting in that particular case, is that the inverse map, from the unit disk to the polygon, has an explicit formula, called the Schwarz-Christoffel formula, which we give below without proof.

Theorem (Schwarz-Christoffel Formula): The functions $z = F(w)$ which map $D_1(0)$ conformally onto polygons with angles $\pi\alpha_k$ with $k \in [\![1, n]\!]$ in counterclockwise order are of the form

$$F(w) = A\int_0^w \prod_{k=1}^{n}(\zeta - w_k)^{-\beta_k}\,d\zeta + B$$

where A and B are complex constants, $\beta_k = 1-\alpha_k$ such that $\sum_{k=1}^{n}\beta_k = 2$, and $(w_k)_{k=1,\dots,n}$ n points on the unit circle.

The $(w_k)_{k=1,\dots,n}$ are called the prevertices: they are the points on the unit circle such that $F(w_k) = z_k$ are the vertices of the polygon.
Now, observe that given two triplets of points on the unit circle $\{w_1, w_2, w_3\}$ and $\{w_1', w_2', w_3'\}$, there exists a Möbius transformation which maps the unit disk to the unit disk, and $\{w_1, w_2, w_3\}$ to $\{w_1', w_2', w_3'\}$. That means that we are always free to choose three of the w_k as we like, provided we are consistent with the ordering of the points.
Hence, for $n \leq 3$, the Schwarz-Christoffel formula can be viewed as an explicit formula for the mapping. A is easily determined by scaling and rotating the polygon appropriately, and B by mapping the origin of the disk to the desired point.
For $n \geq 4$, different choices of prevertices lead to different polygons, all consistent with the angles α_k. This is shown in Figure 5. In general, computing the non-free prevertices for a desired polygon must be done numerically. You may want to have a look at the great book *Schwarz-Christoffel Mapping* by T. Driscoll and L.N. Trefethen, Cambridge University Press, for a description of numerical algorithms and nice examples.

5.2 Generalizing the result: mapping the upper half plane to a polygon

We have seen that the Cayley transform

$$S(z) = \frac{z-i}{z+i}$$

maps the upper half plane to the unit disk. Its inverse is

$$z = S^{-1}(w) = i\frac{1+w}{1-w}$$

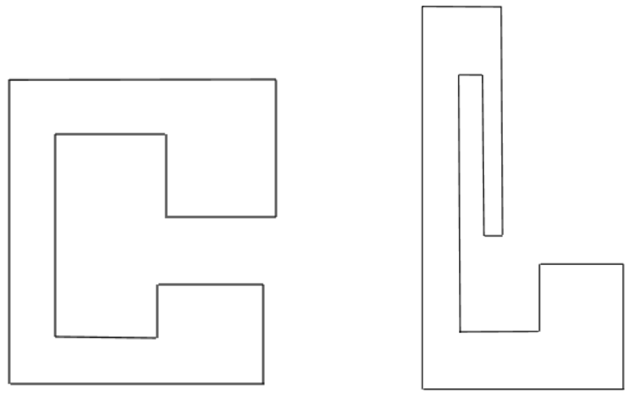

Figure 5: Two possible images of the unit disk for different choices of the prevertices, and the same choice of angles

Composing F with S, we could have a mapping from the upper half plane to polygons. Defining

$$u = S^{-1}(\zeta) = i\frac{1+\zeta}{1-\zeta}$$

$$(\zeta - w_k)^{-\beta_k} = \left(\frac{u-i}{u+i} - w_k\right)^{-\beta_k} = \left(\frac{1-w_k}{u+i}\right)^{-\beta_k}\left[u - i\frac{1+w_k}{1-w_k}\right]^{-\beta_k}$$

We let $v_k := i(1+w_k)/(1-w_k)$, and observe that

$$d\zeta = \frac{2i}{(u+i)^2}du$$

to obtain the general form of the mapping from the upper half plane to polygons:

$$G(z) := F \circ S(z) = 2iA\int_0^z \prod_{k=1}^n \left(\frac{1-w_k}{u+i}\right)^{-\beta_k}(u-v_k)^{-\beta_k}\frac{du}{(u+i)^2} + B$$

Using the fact that $\sum_{k=1}^n \beta_k = 2$, this can be written in the more concise form

$$G(z) = \alpha\int_0^z \prod_{k=1}^n (u-v_k)^{-\beta_k}du + \beta$$

with α and β constants, and β_k as before.

5.3 Examples

•*Mapping the upper half plane to the semi-infinite strip* $-\pi/2 < \Re(w) < \pi/2$, $\Im(w) > 0$
We have $w_1 = -\pi/2$, $w_2 = \pi/2$. Let us choose $v_1 = -1$, $v_2 = 1$. The mapping is given by

$$G(z) = \alpha\int_0^z (u+1)^{-1/2}(u-1)^{-1/2}du + \beta = \alpha\int_0^z \frac{1}{\sqrt{u^2-1}}du + \beta = \alpha' \arcsin z + \beta$$

Setting $G(-1) = -\pi/2$ and $G(1) = \pi/2$ leads to the system

$$\begin{cases} -\frac{\pi}{2}\alpha' + \beta = -\frac{\pi}{2} \\ \frac{\pi}{2}\alpha' + \beta = \frac{\pi}{2} \end{cases}$$

from which we conclude that $\beta = 0$ and $\alpha' = 1$: $f(z) = \arcsin z$ is a map from the upper half plane to the desired semi-infinite strip.
• *Mapping the upper half plane to a rectangle*

Let us assume the rectangle is rotated and translated so that its vertices are $w_1 = -K_1 + iK_2$, $w_2 = -K_1$, $w_3 = K_1$, and $w_4 = K_1 + iK_2$. By symmetry, we choose the prevertices such that $v_1 = -k^{-1/2}$, $v_2 = -1$, $v_3 = 1$, and $v_4 = k^{-1/2}$, where k represents the fact that only 3 points can be freely specified.
We thus have

$$w = G(z) = \beta + \alpha \int_0^z \prod_{k=1}^{4} (u - v_k)^{-1/2} = \alpha \int_0^z \frac{du}{\sqrt{u^2 - \frac{1}{k}}\sqrt{u^2 - 1}}$$

where we have set $\beta = 0$ so that $F(0) = 0$. This is an elliptic integral of the first kind, which can be written in the more concise form

$$w = G(z) = \alpha \int_0^{\arcsin z} \frac{d\theta}{\sqrt{1 - k \sin^2 \theta}}$$

The constant α represents the freedom in rotating and scaling the rectangle, and the constant k must be varied to obtain the desired aspect ratio K_2/K_1 for the rectangle.