The purpose of this lecture is to present a few applications of conformal mappings in problems which arise in physics and engineering.

1 Laplacian operator under an analytic map

Proposition: Consider the real valued function $U(\xi,\eta)$ and the analytic map $w = f(z) = f(x+iy) = \xi(x,y)+i\eta(x,y)$, where ξ and η are real valued functions. The composition $u(x,y) = U(\xi(x,y),\eta(x,y))$ of U with f satisfies

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right) = |f'(z)|^2 \Delta U \tag{1}$$

The proof of this result is straightforward, and left to the reader. It follows directly from the chain rule and the Cauchy-Riemann relations for ξ and η.

A key conclusion from the result above is that if u is harmonic in a region of the z-plane, U is harmonic in a conformally mapped region of the w-plane. As we will now see, this leads to an elegant method for solving Laplace's equation with Dirichlet boundary conditions on certain domains.
Equality (1) is also useful for solving Poisson's equation, as Poisson's equation can be turned into a scaled Poisson's equation on a simpler domain.

2 Applications of conformal mapping

2.1 Harmonic function on the right-half plane

Consider the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 \ , \ x > 0 \\ u(0,y) = h(y) \ \ , \ \ y \in \mathbb{R} \end{cases} \tag{2}$$

The map $w = \xi + i\eta = S(z) = \frac{z-1}{z+1}$ is a one-to-one conformal mapping from the right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ to the unit disk $\{w \in \mathbb{C} : |w| < 1\}$.
If $U(\xi,\eta)$ is harmonic in the unit disk

$$u(x,y) = U(\xi(x,y),\eta(x,y)) = U\left(\frac{x^2+y^2-1}{(x+1)^2+y^2}, \frac{2y}{(x+1)^2+y^2}\right)$$

satisfies $\Delta u = 0$ in the right half plane.

Let $H(\theta) := U(\cos\theta,\sin\theta)$ be the value of U on $C_1(0)$. We need to specify H so that u takes the appropriate boundary value h for $x = 0$. This requires inverting $w = S(z)$:

$$S(z) = \frac{z-1}{z+1} \quad \Rightarrow \quad z = S^{-1}(w) = \frac{w+1}{1-w} \quad \Rightarrow \quad x+iy = \frac{1-\xi^2-\eta^2}{(\xi-1)^2+\eta^2} + i\frac{2\eta}{(\xi-1)^2+\eta^2}$$

Thus, the values of y corresponding to points w on the unit disk are

$$y = \frac{2\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

So the boundary condition on U on the unit disk is

$$H(\theta) = h(\cot\frac{\theta}{2})$$

Using Poisson's integral formula, we can construct a closed form formula for U on $D_1(0)$:

$$\forall z \in D_1(0) \ , \ U(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} h\left(\cot\frac{\theta}{2}\right) d\theta$$

Using the polar coordinates (r, φ) on the unit disk, this can be rewritten as an expression for $U(r, \varphi)$:

$$U(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)} h\left(\cot\frac{\theta}{2}\right) d\theta$$

from which u can be computed directly.

Let us work out a specific example of this type, namely

$$h(y) = \begin{cases} 1 \text{ if } y > 0 \\ 0 \text{ if } y \leq 0 \end{cases}$$

which leads to

$$H(\theta) = \begin{cases} 1 \text{ if } 0 < \theta < \pi \\ 0 \text{ if } \pi \leq \theta < 2\pi \end{cases}$$

The Poisson integral can then be computed exactly, and we find (using numerical software)

$$U(\xi, \eta) = \begin{cases} 1 - \frac{1}{\pi} \arctan\left(\frac{1-\xi^2-\eta^2}{2\eta}\right) \quad , \quad \eta > 0 \\ \frac{1}{2} \quad , \quad \eta = 0 \\ -\frac{1}{\pi} \arctan\left(\frac{1-\xi^2-\eta^2}{2\eta}\right) \quad , \quad \eta < 0 \end{cases}$$

for $\xi^2 + \eta^2 < 1$.
After some straightforward yet lengthy algebra, one then finds the desired answer:

$$u(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{y}{x}\right)$$

Observe that as the problem is stated in (2), it is not well posed: many functions other than the one we just constructed satisfy the equation. For example, $\forall A \in \mathbb{C}$, $v(x, y) = u(x, y) + Ax$ satisfies $\Delta v = 0$ and $v(0, y) = h(y)$. The problem is made well posed, i.e. the solution is made unique by requiring the solution to be bounded at ∞, which is a condition u satisfies.

2.2 Steady heat flow

Fourier's law of heat conduction states that the heat flux $\mathbf{q}$ in a homogeneous isotropic medium Ω of constant thermal conductivity k is given by

$$\mathbf{q} = -k\nabla u$$

where u is the temperature in Ω. For a time-independent temperature u, conservation of energy implies that $\nabla \cdot \mathbf{q} = 0$, i.e. $\Delta u = 0$ in Ω.
When u is specified on the boundary, this is Laplace's equation with Dirichlet boundary conditions.

Let us compute the temperature u in the exterior of two disks $|z - i| < 1$ and $|z + i| < 1$, with Dirichlet conditions $u = 1$ on the first circle, and $u = -1$ on the second circle, as well as $u \to 0$ at ∞.
To solve this problem, observe that the map $w = \frac{1}{z}$ maps our domain of interest to the strip $-\frac{1}{2} < \Im(w) < \frac{1}{2}$.
Therefore, we start by solving

$$\begin{cases} \Delta U(\xi, \eta) = 0 \\ U(\xi, -\frac{1}{2}) = 1 \\ U(\xi, \frac{1}{2}) = -1 \\ U \to 0 \text{ as } (\xi, \eta) \to (0, 0) \end{cases}$$

The unique solution to this problem clearly is

$$U(\xi, \eta) = -2\eta$$

so that

$$u(x, y) = \frac{2y}{x^2 + y^2}$$

is the solution to the original problem.

2.3 Electrostatic potential inside a non-coaxial cable

Gauss' law for the electric field $\mathbf{E}$ due to an electric charge density ρ is given by

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where ϵ_0 is the vacuum permittivity. In the absence of electric charges, the electric field satisfies $\nabla \cdot \mathbf{E} = 0$. Furthermore, for time independent electric fields, Faraday's law of electromagnetics tells us that $\nabla \times \mathbf{E} = \mathbf{0}$, so that $\mathbf{E} = -\nabla\phi$, where ϕ is called the electrostatic potential. In the absence of charges, ϕ satisfies Laplace's equation: $\Delta\phi = 0$.

Let us consider the situation in which one wants to compute the electrostatic potential in the vacuum region between an interior infinitely long cylinder and an exterior infinitely long cylindrical shell. This setup can be viewed as an idealization of a coaxial cable.

The problem as stated is essentially 2D in nature, so we will view it as a small disk inside a larger circle. We consider the general case in which the inner disk is off center: say the disk $|z - \frac{2}{5}| \leq \frac{2}{5}$ inside the circle $|z| = 1$.

ϕ solves the equation $\Delta\phi = 0$ in the region $\{z \in \mathbb{C} : |z - \frac{2}{5}| > \frac{2}{5}, |z| < 1\}$, with boundary conditions $\phi = a$ for $|z| = 1$ and $\phi = b$ for $|z - \frac{2}{5}| = \frac{2}{5}$.

The idea is to reduce, by mapping, the problem to a radially symmetric case, in which the inner disk is centered, so that we know by symmetry that the solution has the form

$$U(r, \theta) = A \ln r + B$$

where we can easily solve for A and B.

Now, we know that

$$S(z) := \frac{z - z_0}{1 - \overline{z_0} z} \quad , \quad |z_0| < 1$$

maps the unit disk to itself. Let us look for z_0 such that the off-axis disk $|z - \frac{2}{5}| = \frac{2}{5}$ is mapped to the centered disk $|z| = R$ for some R.

Let us take $z_0 \in \mathbb{R}$, and try and map 0 and $\frac{4}{5}$ to $-R$ and R. We find

$$\begin{cases} S(0) = -z_0 = -R \\ S\left(\frac{4}{5}\right) = \frac{\frac{4}{5} - z_0}{1 - \frac{4}{5} z_0} = R \end{cases}$$

which gives the quadratic equation

$$-\frac{4}{5} z_0^2 + 2z_0 - \frac{4}{5} = 0$$

which has the solutions $z_0 = \frac{1}{2}$ and $z_0 = 2$. Only the first solution satisfies $|z_0| < 1$, so we consider the mapping

$$S(z) = \frac{2z - 1}{2 - z}$$

which maps our region to the annulus

$$\Omega = \{\frac{1}{2} < |w| < 1\}$$

We can now apply the boundary conditions on u:

$$U(1, \theta) = a \quad \Rightarrow \quad B = a$$

$$U(\frac{1}{2}, \theta) = b \quad \Rightarrow \quad A = \frac{a - b}{\ln 2}$$

Thus, the solution in the mapped domain is

$$U(\xi, \eta) = \frac{a - b}{2 \ln 2} \ln\left(\xi^2 + \eta^2\right) + a$$

from which we can compute the potential in the coaxial:

$$u(x, y) = \frac{a - b}{2 \ln 2} \ln\left[\frac{(2x - 1)^2 + 4y^2}{(x - 2)^2 + y^2}\right] + a$$

2.4 Potential flow

• *Stream function*

Consider an incompressible two-dimensional flow $\mathbf{u} = \langle u_1, u_2 \rangle$, i.e. a 2D flow which satisfies $\nabla \cdot \mathbf{u} = 0$. This equation is automatically satisfied if one defines a function of two variables ψ such that

$$u_1 = \frac{\partial \psi}{\partial y} \quad , \quad u_2 = -\frac{\partial \psi}{\partial x}$$

which can be written in the concise form

$$\mathbf{u} = \nabla \times (\psi \mathbf{e}_z)$$

where $\mathbf{e}_z$ is the third vector of the Cartesian orthonormal basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. In fluid dynamics, ψ is called a stream function. The level curves of ψ, which are called streamlines, are everywhere tangent to the flow.

If in addition, the flow is irrotational, i.e. such that $\nabla \times \mathbf{u} = \mathbf{0}$, we have

$$\nabla \times \mathbf{u} = \mathbf{0} \;\Rightarrow\; \nabla \times (\nabla \times (\psi \mathbf{e}_z)) = \mathbf{0} \quad \Rightarrow \quad \Delta \psi = 0$$

For an irrotational incompressible flow, the stream function ψ satisfies Laplace's equation.

• *Potential flow*

Let us now consider the situation in the opposite direction. If a flow is irrotational, $\nabla \times \mathbf{u} = \mathbf{0}$, there exists a function ϕ of two variables such that

$$\mathbf{u} = \nabla \phi$$

ϕ is called the velocity potential. If, in addition, the flow is incompressible, one way write

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \Delta \phi = 0$$

The velocity potential of an irrotational and incompressible flow satisfies Laplace's equation.

• *The complex potential*

We have just seen that for a two-dimensional irrotational and incompressible flow, there exist two functions of two variables ψ and ψ such that

$$\mathbf{u} = \nabla \phi = \nabla \psi \times \mathbf{e}_z \tag{3}$$

If ϕ and ψ are continuous functions, the complex function

$$w(z) = \phi(x, y) + i\psi(x, y)$$

is analytic because (3) can be seen as Cauchy-Riemann relations for ϕ and ψ. w is called the complex potential.

• *Potential flow around an infinitely long solid body: boundary conditions*

Consider an infinitely long body immersed in a fluid. We can consider the problem to be two-dimensional, just as in the case of the coaxial cable. Imagine that before immersing the object, the flow was $\mathbf{u} = \langle U, V \rangle$. After the body is immersed, flow far away from the body is not disturbed, so the boundary conditions on ϕ and ψ are

$$\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rangle = \langle U, V \rangle \qquad , \qquad \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle = \langle U, V \rangle$$

The boundary condition at a solid static boundary is

$$\mathbf{u} \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the unit normal vector to the boundary of the object. This translates into a Neumann boundary condition for the potential ϕ and a Dirichlet boundary condition for the stream function ψ:

$$\mathbf{n} \cdot \nabla \phi = 0 \qquad , \qquad \mathbf{t} \cdot \nabla \psi = 0$$

where $\mathbf{t}$ is the unit tangent vector to the boundary.

- *Milne-Thomson's circle theorem*

Theorem (Milne-Thomson's circle theorem): Let $f(z)$ be the complex potential for a fluid flow, where all singularities of f lie in a region $|z| > a$. Then, the complex potential

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)}$$

is the complex potential with the same singularities as f in the region $|z| > a$, and $|z| = a$ as a streamline.

Proof: If the singularities of f are in $|z| > a$, the singularities of $f\left(\frac{a^2}{\overline{z}}\right)$ are in

$$\left|\frac{a^2}{\overline{z}}\right| > a \quad \Leftrightarrow \quad |z| < a$$

so the singularities of w in $|z| > a$ are indeed the same as the singularities of f in that region. This proves the first part of the theorem.
Now, for $|z| = a$,

$$w(z) = f(z) + \overline{f\left(\frac{z\overline{z}}{\overline{z}}\right)} = f(z) + \overline{f(z)} \in \mathbb{R}$$

w is purely real, hence $|z| = a$ is the streamline $\psi = 0$.

- *Flow around an infinitely long circular cylinder*

Consider an infinitely long circular cylinder with radius R placed in a flow which was initially uniform in the x-direction: $\mathbf{u} = \langle U, 0\rangle$. The associated complex potential is

$$f(z) = Uz$$

which has one singularity, at infinity.
To obtain the flow in the presence of the circular cylinder, we apply the circle theorem. The complex potential in the presence of the cylinder is

$$w(z) = U\left(z + \frac{R^2}{z}\right)$$

Using polar coordinates, this can be written as

$$w(z) = U\left(re^{i\theta} + \frac{R^2}{r}e^{-i\theta}\right)$$

so that

$$\phi(r,\theta) = U\left(r + \frac{R^2}{r}\right)\cos\theta$$
$$\psi(r,\theta) = U\left(r - \frac{R^2}{r}\right)\sin\theta$$

and the flow in polar coordinates is given by

$$\mathbf{u} = \langle U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta\rangle$$

- *Flow around an infinitely long elliptic cylinder*

Consider the complex potential $w(z) = \phi(x,y) + i\psi(x,y)$ and the forward and inverse conformal maps

$$Z = X + iY = f(z)$$
$$z = x + iy = F(Z)$$

Let

$$W(Z) := w(F(Z)) = \Phi(X,Y) + i\Psi(X,Y)$$

W is an analytic function of Z, and Φ and Ψ satisfy the Cauchy-Riemann equations. They can thus be viewed as the velocity potential and stream function of an irrotational and incompressible flow in the mapped domain. And since $W(f(z)) = w(z)$, a streamline of the original flow in the original domain is mapped to streamline of the mapped flow in the mapped domain.

Consider a slightly different form of the Joukowski map, defined by

$$f(z) = z + \frac{1}{z}$$

From the last lecture, we know that f maps the circle with radius $R > 1$ onto the ellipse

$$X = \left(R + \frac{1}{R}\right)\cos\theta$$
$$Y = \left(R - \frac{1}{R}\right)\sin\theta$$

and the exterior of the circle is mapped onto the exterior of the ellipse. The inverse of the Joukowski map is

$$z = F(Z) = \frac{Z}{2} + \sqrt{\frac{Z^2}{4} - 1}$$

The flow past the elliptic cylinder is thus given by the following complex potential:

$$W(Z) = U\left(\frac{Z}{2} + \sqrt{\frac{Z^2}{4} - 1}\right) + UR^2\left(\frac{Z}{2} - \sqrt{\frac{Z^2}{4} - 1}\right) = U\left[(1+R^2)\frac{Z}{2} + (1-R^2)\sqrt{\frac{Z^2}{4} - 1}\right]$$

It is clear that this solution satisfies the proper boundary condition at the surface of the cylinder, since a streamline is mapped to a streamline. What about the boundary condition at infinity?
We may write

$$\frac{dW}{dZ} = \frac{dw}{dz}\frac{dF}{dz} = \frac{1}{f'(z)}\frac{dw}{dz}$$

In our case, $f'(z) = 1 - 1/z^2 \to 1$ as $z \to \infty$, so the uniform flow at infinity is indeed mapped to the same uniform flow at infinity.