We often encounter analytic functions defined by their power series expansion on a certain disk of convergence, by an integral representation, or by an infinite sum, as was the case for the gamma and zeta functions in Lecture 13. As we noticed empirically in that lecture, it may sometimes be possible to extend such functions to analytic functions on larger domains. This extension process is called *analytic continuation*, and is the focus of the present lecture.

Analytic continuation can be used to construct analytic functions from multiple-valued functions, such as ln and n^{th} roots. This is best framed and understood in the context of functions defined on Riemann surfaces. However, due to time constraints, we will follow Rudin's treatment in this lecture, and omit the connection with Riemann surfaces in our presentation.

1 Singular points

1.1 Definition

Definition: Consider an open disk D, a function f which is analytic on D, and z_0 a point on the boundary ∂D of the disk. z_0 is called a *regular point* of f if there exists a disk D^0 centered in z_0 and an analytic function g on D^0 such that $\forall z \in D \cap D^0$, $g(z) = f(z)$.
Any point which is on the boundary of D and which is not a regular point of f is called a *singular point* of f.

Note: The definition implies that the set of regular points of f is an open subset of the boundary ∂D of D.

1.2 Singular points and radius of convergence

Theorem: Suppose f is analytic on $D_R(z_0)$ and the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has radius of convergence R. Then f has at least one singular point on the circle $C_R(z_0)$.

Proof: Without loss of generality, let us take f to be analytic on $D_1(0)$, and the power series $\sum_{n=0}^{\infty} c_n z^n$ with radius of convergence 1.
Suppose that every point of $C_1(0)$ is a regular point of f. Then we will show that the radius of convergence of the series is greater than 1.
Since $C_1(0)$ is compact, one can cover it with N open disks $(D^i)_{i=1,\ldots,N}$ whose centers are on $C_1(0)$. From our assumption, we can choose these disks such that there exist analytic functions g_i, $i = 1, \ldots, N$ on each of these disks such that $g_i(z) = f(z)$ on $D^i \cap D_1(0)$.
Now, let

$$\Omega_{ij} := D^i \cap D^j \cap D_1(0)$$

If $D^i \cap D^j \neq \{\emptyset\}$, then $\Omega_{ij} \neq \{\emptyset\}$ and $g_i(z) = f(z) = g_j(z)$ $\forall z \in \Omega_{ij}$. By the identity theorem, this means that

$$g_i(z) = g_j(z) \ \ , \ \ \forall z \in D^i \cap D^j$$

We can therefore define an analytic function h on $\Omega := D_1(0) \cup D^1 \cup \ldots \cup D^N$ by

$$h(z) = \begin{cases} f(z) \text{ if } z \in \ D_1(0) \\ g_i(z) \text{ if } z \in D^i \end{cases}$$

Ω is open, and $\overline{D_1(0)} \subset \Omega$ so there exists $r > 0$ such that $D_{1+r} \subset \Omega$. By the identity theorem, h has the power series $\sum_{n=0}^{\infty} c_n z^n$ in $D_{1+r}(0)$, so the radius of convergence of the series is at least $1 + r$ $\square$

1.3 Natural boundary of an analytic function

In what follows, we focus once again on the unit disk $D_1(0)$, without loss of generality.

Definition: If f is analytic on $D_1(0)$ and if every point of $C_1(0)$ is a singular point of f, then $C_1(0)$ is said to be the *natural boundary* of f.
In that case, f has no analytic extension to any open connected set which contains $D_1(0)$.

Here is an example of such a function. Consider

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

with radius of convergence 1. f is unbounded in the neighborhood of $z = 1$, and since

$$\forall z \in D_1(0)\ ,\ f(z^2) = f(z) - z$$

f is also unbounded in the neighborhood of $e^{i\pi} = -1$. Repeating the process, f is also unbounded in the neighborhood of $e^{i\pi/2}$ and $e^{3i\pi/2}$. More generally, f is unbounded in the neighborhood of $z_{mn} = e^{2\pi i m/2^n}$, where m and n are positive integers.
The $(z_{mn})_(m,n) \in \mathbb{N}^2$ are a dense subset of $C_1(0)$, and since the set of singular points of f is a closed set, $C_1(0)$ is the natural boundary of f.

In what follows, we give a sufficient condition for a power series to have $C_1(0)$ as its natural boundary. We will however not provide a proof, which can be found in Rudin's *Real and Complex Analysis*. The theorem, due to Hadamard, is as follows.

Theorem: Suppose k is a positive integer and $(p_n)_{n\in\mathbb{N}^*}$ a sequence of positive integers such that

$$\forall n \in \mathbb{N}^*\quad ,\quad p_{n+1} > \left(1 + \frac{1}{k}\right) p_n$$

and the power series $f(z) = \sum_{n=1}^{\infty} c_n z^{p_n}$ has radius of convergence 1. Then f has $C_1(0)$ as its natural boundary.

The series $\sum_{n=0}^{\infty} z^{2^n}$ we just considered is an illustration of this result. A key motivation for bringing up this theorem here is to study a power series which will help dispel misconceptions about the smoothness and behavior of functions in the neighborhood of singular points as defined earlier in the lecture.
Consider the function f defined by the power series

$$f(z) = \sum_{n=0}^{\infty} c_n e^{-\sqrt{n}} z^n$$

where $c_n = 1$ if n is a power of 2, and $c_n = 0$ otherwise. The radius of convergence of the power series is 1 since

$$\limsup_{n\to\infty} |c_n e^{-\sqrt{n}}|^{1/n} = 1$$

Furthermore, by the theorem above we know that f has $C_1(0)$ as its natural boundary.
Now, observe that the k^{th} derivative of f is given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)c_n e^{\sqrt{n}} z^{n-k}$$

We can see that $f^{(k)}$ is uniformly continuous on $\overline{D_1(0)}$ since the series above converges uniformly on that closed disk. We conclude that the restriction of f to $C_1(0)$ is infinitely differentiable (as a function of one variable parametrizing the curve, e.g. arc length).

2 Analytic continuation along curves

2.1 Function elements

Definition: A *function element* is an ordered pair (f, D), where D is an open disk, and f an analytic function on D. We say that the functions elements (f_0, D^0) and (f_1, D^1) are *direct continuations* of each other, which we write $(f_0, D^0) \sim (f_1, D^1)$ if

1. $D^0 \cap D^1 \neq \{\emptyset\}$
2. $\forall z \in D^0 \cap D^1,\ f_0(z) = f_1(z)$

2.2 Chains of disks

Definition: A *chain* is a finite sequence $(D^i)_{i=1,\dots,N}$ of disks such that

$$\forall i \in [\![1, n]\!]\ ,\ D^{i-1} \cap D^i \neq \{\emptyset\}$$

Let us take some ordered pair (f_0, D^0). If for all $i \in [\![1,n]\!]$, there exists (f_i, D^i) such that $(f_i, D^i) \sim (f_{i-1}, D^{i-1})$, then (f_n, D^n) is called an *analytic continuation* of (f_0, D^0) along the chain $\mathcal{C} = (D^i)_{i=1,\dots,N}$.

By the identity theorem combined with an induction, f_n is uniquely determined by f_0.

Observe however that $\sim$ is not transitive: if (f_n, D^n) is called an analytic continuation of (f_0, D^0) along the chain $\mathcal{C}$, it may still be that $(f_0, D^0) \nsim (f_n, D^n)$.
A typical example illustrating the non-transitivity of $\sim$ is as follows. Consider the unit disks D^0, D^1 and D^2 centered at 1, ω, and ω^2, where ω is the cubic root of unity. Choose analytic functions f_j on D^j such that $\forall j \in \{1, 2, 3\}$, $\forall z \in D^j$, $f_j^2(z) = z$. We know from Lecture 9 that this is indeed possible.
Now, we construct the f_j such that $(f_0, D^0) \sim (f_1, D^1) \sim (f_2, D^2)$. Consider

$$f_j(z) = e^{ij\frac{\pi}{3}} f_0\left(ze^{-ij\frac{2\pi}{3}}\right)$$

where $\forall z \in D^0$, $f_0(z) = \exp\left(\frac{1}{2}\text{Ln}z\right)$. Observe that

$$\forall z \in D^j\ ,\ f_j^2(z) = e^{-2ij\frac{\pi}{3}} f_0^2\left(ze^{-ij\frac{2\pi}{3}}\right) = z$$

as desired. Furthermore, $\forall z \in D^0 \cap D^1$, $w = e^{-2\pi i/3}z$ lies in D^0 and

$$f_0(w) = e^{-i\pi/3} f_0(z) \ \Leftrightarrow\ f_0(z) = e^{i\pi/3} f_0\left(ze^{-i\frac{2\pi}{3}}\right) = f_1(z)$$

Likewise, for $z \in D^1 \cap D^2$, both $w_1 = e^{-4\pi i/3}z$ and $w = e^{-2\pi i/3}z$ are in D^0, and $w_1 = e^{-2\pi i/3}w$, so

$$f_0(w_1) = e^{-i\pi/3} f_0(w) \ \Leftrightarrow f_0(e^{-2\pi i/3}z) = e^{i\pi/3} f_0(e^{-4\pi i/3}z) \ \Leftrightarrow\ e^{-i\pi/3} f_1(z) = e^{i\pi/3} e^{-2\pi i/3} f_2(z)$$

so that $f_1(z) = f_2(z)$.
However, $\forall z \in D^0 \cap D^2$, $w = e^{-4\pi i/3}z$ is in D^0 so

$$f_0(w) = e^{i\pi/3} f_0(z) \ \Leftrightarrow f_0(z) = e^{-i\pi/3} f_0(e^{-4\pi i/3}z) = e^{-i\pi/3} e^{-2i\pi/3} f_2(z) = -f_2(z)$$

from which we see that $(f_0, D^0) \nsim (f_2, D^2)$.

2.3 Continuation along curves

Definition: A chain $\mathcal{C} = (D^i)_{i=1,\dots,n}$ is said to cover a parameterized curve $\gamma : t \in [0, 1] \to \gamma(t)$ if there are numbers $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ such that $\gamma(t_0)$ is the center of D^0, $\gamma(t_n)$ is the center of D^n, and $\forall i \in [\![1,n-1]\!]$, $\gamma([t_i, t_{i+1}]) \subset D^i$.

If (f_0, D^0) can be continued along this chain to (f_n, D^n), (f_n, D^n) is said to be an *analytic continuation* of (f_0, D^0) along γ.

We will show that this analytic continuation, if it exists, is unique.

Theorem: If (f, D) is a function element and if γ is a curve which starts at the center of D, then (f, D) admits at most one analytic continuation along γ.

In other words, what we want to prove is the following. If γ is covered by a chain $\mathcal{C}_1 = (D^i)_{i=1,\ldots,n}$ and a chain $\mathcal{C}_2 = (B^i)_{i=1,\ldots,m}$, where the B^i are also disks, and such that $D^0 = B^0$, and if (f, D^0) can be analytically continued along $\mathcal{C}_1$ to (g_n, D^n) while (f, B^0) can be analytically continued along $\mathcal{C}_2$ to (h_m, B^m), then $h_m = g_n$ in $D^n \cap B^m$.

To prove this theorem, we start with a simple transitivity result for the relation $\sim$:
Suppose $D^0 \cap D^1 \cap D^2 \neq \{\emptyset\}$, and $(D^0, f_0) \sim (D^1, f_1)$, and $(D^1, f_1) \sim (D^2, f_2)$. Then $(D^0, f_0) \sim (D^2, f_2)$. The proof of this result is straightforward, and follows directly from the identity theorem. It is left to the reader. We are now ready for the proof of the theorem itself.

We take $\mathcal{C}_1$ and $\mathcal{C}_2$ as above. There are numbers $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1 = t_{n+1}$ and $0 = s_0 < s_1 < \ldots < s_{m-1} < s_m = 1 = s_{m+1}$ such that

$$\gamma([t_i, t_{i+1}]) \subset D^i \ , \ \ \gamma([s_j, s_{j+1}]) \subset B^j \quad , \ 0 \leq i \leq n \ \ , \ 0 \leq j \leq m$$

By hypothesis,

$$\forall i \in [\![0, n-1]\!] \ , \ \forall j \in [\![0, m-1]\!] \ \ , \ (g_i, D^i) \sim (g_{i+1}, D^{i+1}) \ \ , \ \ (h_j, B^j) \sim (h_{j+1}, B^{j+1})$$

We claim that $\forall i \in [\![0, n]\!]$, $\forall j \in [\![0, m]\!]$, if $[t_i, t_{i+1}] \cap [s_j, s_{j+1}] \neq \{\emptyset\}$, $(g_i, D^i) \sim (h_j, B^j)$. Indeed, let us assume there are pairs (i, j) for which this is wrong. Choose among these pairs the one corresponding to the minimal $i + j$. By hypothesis, $i + j > 0$.
Suppose $t_i \geq s_j$. Then it must be that $i \geq 1$, and if the intersection between $[t_i, t_{i+1}]$ and $[s_j, s_{j+1}]$ is nonempty, we can say that $\gamma(t_i) \in D^{i-1} \cap D^i \cap B^j$. Now, since we chose the minimal pair (i, j), it must be that $(g_{i-1}, D^{i-1}) \sim (h_j, B^j)$. Furthermore, by hypothesis $(g_{i-1}, D^{i-1}) \sim (g_i, D^i)$. Our transitivity result above then tells us that $(g_i, D^i) \sim (h_j, B^j)$, which is a contradiction.
We can follow the same logic for the case $s_j \geq t_i$, and end up with the same contradiction. We conclude that

$$[t_i, t_{i+1}] \cap [s_j, s_{j+1}] \neq \{\emptyset\} \ \Rightarrow \ (g_i, D^i) \sim (h_j, B^j) \ , \forall i \in [\![0, n]\!] \ , \ \forall j \in [\![0, m]\!]$$

In particular, $(g_n, D^n) \sim (h_m, B^m)$ $\square$

3 The monodromy theorem

3.1 One-parameter family of curves

__**Definition**__: Consider $(z_0, z_1) \in \mathbb{C}^2$ and a continuous mapping $\varphi : [0, 1]^2 \to \mathbb{C}$ such that $\forall t \in [0, 1]$, $\varphi(0, t) = z_0$ and $\varphi(1, t) = z_1$. The family of curves γ_t defined by

$$\gamma_t(s) := \varphi(s, t) \quad , \ 0 \leq s \leq 1 \ , \ \ 0 \leq t \leq 1$$

forms a *one-parameter family* $(\gamma_t)_{t \in [0,1]}$ of curves from z_0 to z_1.

Theorem: Let $(\gamma_t)_{t \in [0,1]}$ be a one-parameter family of curves from z_0 to z_1, D be an open disk with center z_0. Let us assume that the function element (f, D) admits an analytic continuation along each γ_t, to an element (g_t, D^t). Then $g_1 = g_0$ in the sense that $(g_0, D^0) \sim (g_1, D^1)$, with D^0 and D^1 two open disks with center z_1.

Proof: Let t such that $0 \leq t \leq 1$. By hypothesis, there is a chain $\mathcal{C} = (D^i)_{i=1,\ldots,n}$ which covers γ_t, with $D^0 = D$. That is to say, there are numbers $0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = 1$ such that $\forall i \in [\![0, n-1]\!]$, $E_i := \gamma_t([s_i, s_{i+1}]) \subset D^i$.
Now, consider the complements $\left(D^i\right)^c$ of D^i in $\mathbb{C}$. $\exists \epsilon > 0$ such that $\forall i \in [\![0, n-1]\!]$, $d(E_i, (D^i)^c) > \epsilon$.
φ is a continuous mapping on $[0, 1]^2$, so it is uniformly continuous on $[0, 1]^2$, and for the ϵ chosen above, $\exists \delta > 0$ such that

$$\forall (s, u) \in [0, 1]^2 \ , \ \ |u - t| < \delta \ \Rightarrow \ |\gamma_t(s) - \gamma_u(s)| < \epsilon$$

Hence, if u is such that $|u-t|<\delta$, $\mathcal{C}$ covers γ_u. From the theorem in Section 2, we can then say that $g_t = g_u$, since they are obtained by continuation of (f, D) along the same chain.
We have therefore shown that any $t \in [0,1]$ is covered by a segment $I_t := u \in [0,1] \ : \ |u-t| < \delta$ on which $g_u = g_t \ \forall u \in I_t$. Using the compactness and connectedness of $[0,1]$, it is then clear that one can incrementally cover $[0,1]$ by finitely many I_t until we get the desired equality $g_1 = g_0$ in the sense that $(g_0, D^0) \sim (g_1, D^1)$
□

The theorem we just proved is very useful for proving the Monodromy Theorem, which we discuss next.

3.2 Unrestricted continuation

Let z_0 and z_1 be two points in an open connected set Ω, and Γ_0 and Γ_1 two curves in Ω from z_0 to z_1. Γ_0 and Γ_1 are Ω-homotopic if there is a one-parameter family $(\gamma_t)_{t\in[0,1]}$ of curves from z_0 to z_1 such that each γ_t is a curve in Ω, and $\gamma_0 = \Gamma_0$ and $\gamma_1 = \Gamma_1$.

Definition: Consider an open connected set Ω and a function element (f, D), with $D \subset \Omega$. We say that (f, D) admits *unrestricted continuation* in Ω if (f, D) can be analytically continued along every curve in Ω which starts at the center of D.

Theorem (Monodromy Theorem): Suppose the function element (f, D) admits unrestricted continuation in an open connected set Ω. Then

(i) If Γ_0 and Γ_1 are Ω-homotopic curves from z_0 to z_1, where z_0 is the center of D, then the continuation of (f, D) along Γ_0 coincides with its continuation along Γ_1

(ii) If Ω is simply connected, there exists a function g which is analytic on Ω such that $\forall z \in D$, $g(z) = f(z)$

Proof: The first item is a direct consequence of the theorem in Section 3.1. There is nothing more to prove.

Let us now look at (ii). Since Ω is simply connected, any two curves in Ω from z_0 to z_1 are Ω-homotopic, as we know from the usual definition of simple-connectedness, and as we have proved from Ahlfors' definition of simple-connectedness in Lecture 15. Thus, if z_0 is the center of D, we can use (i) to say that all analytic continuations of (f, D) to z_1 lead to the same element (g_1, D^1), where D^1 is a disk with center z_1.
Consider another point z_2, and an analytic continuation (g_2, D^2), where D^2 is a disk with center z_2. Observe that (g_2, D^2) can be obtained by analytically continuing (f, D) to (g_1, D^1), and then along the line segment from z_1 to z_2. Hence

$$\forall z \in D^1 \cap D^2 \ , \ g_1(z) = g_2(z)$$

We may thus define the global function g on Ω by

$$g(z) := g_i(z) \quad , \quad z \in D^i$$

which gives the desired analytic continuation of f to Ω.