# 1 Brief review of limits, continuity, and differentiation

With the stereographic projection, we have just seen, without clearly stating it, our first example of a function which takes real numbers as inputs, and outputs a complex number.

As we move on to studying functions and their properties, 4 cases may in principle be considered: real functions of real variables, real functions of complex variables, complex functions of real variables, and complex functions of complex variables. Fortunately, the vast majority of concepts we will apply to functions can be defined in the same way in the 4 cases. This is because the concept of limit can be defined in the same way in all 4 cases.

## 1.1 Limits

**Definition**: A function $f$ has the limit $L$ ($L$ finite) as $z$ tends to $z_0$, written $\lim_{z\to z_0} f(z) = L$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ for all $z$ such that $|z - z_0| < \delta$.

You can observe that this indeed agrees with the definition you have all seen for real functions of real variables. If the input $z$ is complex, or the output $f(z)$ is complex, what used to be the absolute value should now be understood as the modulus.

Similar definitions are easily constructed for the cases in which $L$ is infinite, as you have done for real variables.

• As we have seen in the previous lecture, for two complex numbers $z_1$ and $z_2$, $|z_1 z_2| = |z_1||z_2|$ and $|z_1 + z_2| \leq |z_1| + |z_2|$ so recalling the proofs in the real variables case, we easily see that the limit laws (sum law, product law) also hold in the complex case.

• Since $|z| = |\overline{z}|$ for any $z \in \mathbb{C}$, if $\lim_{z\to z_0} f(z) = L$, then $\lim_{z\to z_0} \overline{f(z)} = \overline{L}$

• Combining the previous two results,

$$\begin{cases} \lim_{z\to z_0} \Re(f(z)) = \Re(L) \\ \lim_{z\to z_0} \Im(f(z)) = \Im(L) \end{cases} \tag{1}$$

which can be seen as an alternative way of defining the limit of $f(z)$.

## 1.2 Continuity

**Definition**: A function $f$ is continuous at $z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$.

This is again the same definition as in the case of real variables so we know that we could easily prove that the sum and product of continuous functions are continuous functions.

From the definition of the limit, we can also conclude that if $f$ is continuous at $z_0$, then so is $\overline{f}$, $\Re(f)$ and $\Im(f)$.

## 1.3 Derivatives

**Definition**: A function $f$ is differentiable at $z_0$ if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This number is called the *derivative of $f$ at $z_0$*, written

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Once more, the definition is standard. However, depending on the case considered – real function or complex function, real variable or complex variable – the existence of a derivative can have far-reaching consequences regarding the properties of the function.

• Let us start with the simplest case: a complex function $f$ of a real variable $x$. One may write

$$f(x) = u(x) + iv(x)$$

$f$ has a derivative $f'(x_0)$ at $x_0$ if and only if $u$ and $v$ are differentiable at $x_0$, and

$$f'(x_0) = u'(x_0) + iv'(x_0)$$

• Consider now a *real* function $f$ of a *complex* variable $z$. If $f$ is differentiable at $z_0$, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

This is in particular true along the horizontal line $z = z_0 + h$ , $h \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

$f'(z_0)$ is therefore a real number.
But $z$ can also approach $z_0$ along the vertical line $z = z_0 + ih$, with $h \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{ih} = f'(z_0)$$

from which we conclude that $f'(z_0)$ is also purely imaginary, *and thus zero.*

We proved the following result: *If a real function of a complex variable is differentiable at a point, then its derivative is zero at this point.*

• For complex functions of complex variables, differentiability has fundamental consequences for the properties of the function. We now devote the next section (and many more in the remainder of this course) to this crucial case.

# 2 Analytic functions

## 2.1 Definition

**Definition**: A complex function $f$ of a complex variable $z$ is *analytic at* $z_0$ (or *holomorphic at* $z_0$) if

$$f'(z_0) = \lim_{z \to 0} \frac{f(z + z_0) - f(z_0)}{z}$$

exists.

First consequence: if a function $f$ is analytic at $z_0$, it is continuous at $z_0$. Indeed,

$$\lim_{z \to 0} [f(z_0 + z) - f(z_0)] = \lim_{z \to 0} z f'(z_0) = 0$$

Second consequence: If $f$ and $g$ are two functions that are analytic at $z_0$, then so is $f + g$, and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$
- So is also their product $fg$, and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
- So is also their quotient $f/g$ provided $g(z_0) \neq 0$, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

- If $f$ is analytic at $z_0$, and $g$ is analytic at $w_0 = f(z_0)$, then $g \circ f$ is analytic at $z_0$, and $(g \circ f)'(z_0) = f'(z_0)g'(f(z_0))$

Example:
It is easy to verify that $f(z) = z$ is analytic on $\mathbb{C}$, and that $g(z) = 1 + z^2$ is analytic on $\mathbb{C}$ and nonzero on $\mathbb{C} \setminus \{-i, i\}$. Hence $h(z) = z/(1 + z^2)$ is analytic on $\mathbb{C} \setminus \{-i, i\}$.

## 2.2 Cauchy-Riemann equations

For any $z \in \mathbb{C}$, let us write $z = x + iy$, with $(x, y) \in \mathbb{R}^2$, and $f(z) = u(x, y) + iv(x, y)$, where $u$ and $v$ are real valued functions.

If $f$ is analytic at $z_0$, then $f'(z_0) = \lim_{z \to 0} (f(z_0 + z) - f(z_0))/z$ exists, independently of the path that $z$ follows towards 0 in the complex plane. In particular,

$$
\begin{aligned}
f'(z_0) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \left[ \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \right] \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \qquad (2)
\end{aligned}
$$

But we can also write

$$
\begin{aligned}
f'(z_0) &= \lim_{\substack{z \to 0 \\ z \in i\mathbb{R}}} \frac{f(z_0 + z) - f(z_0)}{z} \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x_0, y_0 + h) + iv(x_0, y_0 + h) - u(x_0, y_0) - iv(x_0, y_0)}{ih} \\
&= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \qquad (3)
\end{aligned}
$$

Equating (2) and (3), we must have

$$
\begin{cases}
\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\
\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}
\end{cases} \qquad (4)
$$

These are the well-known *Cauchy-Riemann equations*. **If a function $f$ is holomorphic, then its real and imaginary parts satisfy the Cauchy-Riemann equations**.

Conversely, let us assume $f(z) = u(x, y) + iv(x, y)$ with $u$ and $v$ real valued functions which have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations. Then, one may expand

$$
\begin{aligned}
u(x_0 + h, y_0 + k) &= u(x_0, y_0) + \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + \epsilon_1 h + \epsilon_2 k \\
v(x_0 + h, y_0 + k) &= v(x_0, y_0) + \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + \epsilon_3 h + \epsilon_4 k
\end{aligned} \qquad (5)
$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, $\epsilon_3 \to 0$, $\epsilon_4 \to 0$ as $(h, k) \to (0, 0)$. Hence,

$$
\lim_{h + ik \to 0} \frac{f(z_0 + h + ik) - f(z_0)}{h + ik} = \lim_{(h,k) \to (0,0)} \frac{\frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + i(\frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k)}{h + ik}
$$

Using the Cauchy-Riemann equations, this becomes

$$
\lim_{(h,k) \to (0,0)} \frac{(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x})(h + ik)}{h + ik} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
$$

We conclude that $f$ is analytic.

We have just proved the following theorem:

**Theorem:** $f(z) = u(x,y) + iv(x,y)$ is analytic with continuous derivative $f'(a)$ at $a$ iff $u(x,y)$ and $v(x,y)$ have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations.

The results above give us explicit ways to write $f'(z)$ in terms of the real and imaginary parts of $f$:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic on all of $\mathbb{C}$ since

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and $u_x$, $u_y$, $v_x$, $v_y$ are clearly continuous on $\mathbb{R}^2$.

$$f'(z) = 2x + 2iy = 2z$$

## 2.3 Harmonic functions

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function. Then $u$ and $v$ satisfy the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Let us take for granted, for the time being, that $u$ and $v$ have continuous higher order partial derivatives (we will prove this later in this course). We can write

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0 \end{aligned}$$

In other words, $\Delta u = 0$, $\Delta v = 0$. The real and imaginary parts of an analytic function satisfy Laplace's equation. They are *harmonic functions*.

If two harmonic functions $u$ and $v$ satisfy the Cauchy-Riemann equations, then we say that $v$ is a *conjugate harmonic function of* $u$.

Example: It is easy to see that $f(z) = z^3$ is analytic on $\mathbb{C}$. We can write $u(x,y) = x^3 - 3y^2x$, $v(x,y) = 3x^2y - y^3$, and compute

$$\Delta u = 6x - 6x = 0 \quad , \quad \Delta v = 6y - 6y = 0$$