1 Polynomials

1.1 Construction

$f : z \mapsto z$ is analytic on all of $\mathbb{C}$ since its real and imaginary parts satisfy the Cauchy-Riemann relations and have continuous first-order partial derivatives for all $(x, y) \in \mathbb{R}^2$.

A function f which is analytic on all of $\mathbb{C}$ is called an *entire* function. $f(z) = z$ is an entire function.

Considering our result for the sum and product of analytic functions, this means that for $(a_0, a_1, \ldots, a_N) \in \mathbb{C}^{N+1}$, the polynomial

$$P(z) = \sum_{i=0}^{N} a_i z^i$$

is also an entire function. From our result for the derivative of the product of functions, for any $a \in \mathbb{C}$, we may write

$$P'(a) = \sum_{i=1}^{N} i a_i z^{i-1}$$

1.2 Elementary results

• In this course, we will soon prove the *fundamental theorem of algebra*: Every polynomial P of a complex variable has a root.

Using this result without proof for the time being, we can say that any polynomial of degree N can be written as

$$P(z) = a_N(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_N)$$

where the α_i may or may not be distinct.

The α_i's are called the *zeros* of P. If k of the α_i coincide, we say that α_i is a *zero of order* k. A zero of order 1 is called a *simple zero*. If α_i is a *zero of order* k, one may write $P(z) = (z - \alpha_i)^k P_k(z)$, with P_k a polynomial such that $P_k(\alpha_i) \neq 0$. Thus, $P(\alpha_i) = P'(\alpha_i) = \ldots = P^{(k-1)}(\alpha_i) = 0$, but $P^{(k)}(\alpha_i) \neq 0$: the order of a zero equals the order of the first nonvanishing derivative.

• **Gauss-Lucas theorem**: The smallest convex polygon that contains the zeros of P also contains the zeros of P'.

Proof: Let us consider a zero a of P'. If a is also a zero of P, there is nothing to prove. If a is not a zero of P, we can write

$$\frac{P'(a)}{P(a)} = \sum_{i=1}^{N} \frac{1}{a - \alpha_i} = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \frac{\overline{a} - \overline{\alpha_i}}{|a - \alpha_i|^2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \frac{a - \alpha_i}{|a - \alpha_i|^2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \frac{a}{|a - \alpha_i|^2} = \sum_{i=1}^{N} \frac{\alpha_i}{|a - \alpha_i|^2}$$

$$\Leftrightarrow a = \frac{1}{\sum_{i=1}^{N} \frac{1}{|a-\alpha_i|^2}} \sum_{i=1}^{N} \frac{\alpha_i}{|a - \alpha_i|^2}$$

a can thus be viewed as the barycenter of the roots of α_i of P with positive coefficients that sum to one (i.e. a convex combination of all the roots of P). This completes our proof.

2 Rational functions

2.1 Construction

Consider two polynomials P and Q which do not have common zeros. Then the rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

is analytic away from the zeros of Q.
The zeros of Q are called *poles* of R, and the *order of a pole* is equal to the order of the corresponding zero of Q.

2.2 Counting poles and zeros

Rational functions are often defined over the extended complex plane $\hat{\mathbb{C}}$. To do this, one considers the function $R_1(z) = R\left(\frac{1}{z}\right)$.

If $R_1(0) = 0$, R *has a zero* at ∞.
If $R_1(0) = \infty$, R *has a pole* at ∞.

Example: Let $R(z) = z$, $R_1(z) = 1/z$, so R has a pole of order 1 at ∞.

Now, let us use R_1 to count the number of poles and of zeros of an arbitrary rational function

$$R(z) = \frac{a_N z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0}{b_M z^M + b_{M-1} z^{M-1} + \ldots + b_1 z + b_0} \quad \Rightarrow \quad R_1(z) = z^{M-N} \frac{a_N + a_{N-1} z + \ldots + a_1 z^{N-1} + a_0 z^N}{b_M + b_{M-1} z + \ldots + b_1 z^{M-1} + b_0 z^M}$$

- If $M > N$, then R has a zero of order $M - N$ at ∞
The total number of zeros (counting the order of each zero) is $M - N + N = M = \max(N, M)$
The total number of poles is $M = \max(N, M)$

- If $N > M$, then R has a pole of order $N - M$ at ∞
The total number of zeros is $N = \max(N, M)$.
The total number of poles is $M + N - M = N = \max(N, M)$.

- If $N = M$, then

$$R_1(0) = \frac{a_N}{b_M} \begin{cases} \neq 0 \\ \neq \infty \end{cases}$$

R has neither a pole nor a zero at ∞.
The total number of zeros is $N = M = \max(N, M)$.
The total number of poles is $M = N = \max(M, N)$.

Theorem: The total number of poles and zeros of a rational function R is the same. That number is called the order p of a rational function.
For any $a \in \mathbb{C}$, the equation $R(z) = a$ has p roots, since $\tilde{R}(z) = R(z) - a$ is a rational fraction with the same poles as R.

We conclude this section with an example of rational function which we will study in detail later in this course: *linear fractions*, i.e. rational functions of order 1:

$$R(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. Such rational function is often called *linear fractional transformation*, or *Möbius transform.*

From the theorem, we know that for any $w \in \mathbb{C}$, $R(z) = w$ has a unique solution, which is

$$z = R^{-1}(w) = \frac{dw - b}{-cw + a}$$

2.3 Partial fractions

• Let us consider a rational function $R(z) = \frac{P(z)}{Q(z)}$ such that R has a pole at ∞. By Euclidean division of P by Q, one can write

$$R(z) = G(z) + H(z)$$

where G is a polynomial without constant term and H is a rational function for which the degree of the numerator is at most equal to that of the denominator, so that $H(\infty)$ is finite.

The degree of G is the order of the pole of R at ∞; G is called *the singular part of R at ∞*.

• Now, let the *finite* poles of R be $\beta_1, \beta_2, \dots, \beta_k$, and consider the function $R_j(\zeta) = R(\beta_j + \frac{1}{\zeta})$. R_j is a rational function with a pole at ∞. We can decompose it as

$$R_j(\zeta) = G_j(\zeta) + H_j(\zeta)$$

just as we did for R before, with H_j finite at ∞.

By back substitution, we then have

$$R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right)$$

where G_j is a polynomial in $1/(z-\beta_j)$ without constant term, and is the singular part of R at β_j, and H_j is finite for $z = \beta_j$.

• Consider now the function

$$F(z) := R(z) - G(z) - \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right)$$

F can only have poles at $(\beta_j)_{j=1,\dots,k}$ and at ∞. Furthermore, by construction F is finite at β_j, $1 \le j \le k$ and at ∞, since H is finite at ∞ and the H_j, $1 \le j \le k$ are finite at their respective β_j. F is a rational function which is finite everywhere: it must be a constant.

Absorbing this constant inside G, which we then call $\tilde{G}$, one may write the desired expression:

$$R(z) = \tilde{G}(z) + \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right) \tag{1}$$

The construction above demonstrates rigorously that every rational function has a representation by partial fractions. You have certainly already used partial fractions as a technique to compute integrals of rational functions.

3 Power series

The most natural way to define the next class of analytic functions (exp, cosh, sinh, cos, sin,...) is through power series. We start by looking at some key properties of power series.

3.1 Radius of convergence

Theorem: Consider the sequence $(c_n)_{n\in\mathbb{N}}$ of complex numbers. There exists $R \in \mathbb{R}^+ \cup \{+\infty\}$ such that

$$\begin{cases} \displaystyle\sum_{c=0}^{\infty} c_n z^n \text{ converges absolutely for } 0 \le |z| < R \\ \displaystyle\sum_{c=0}^{\infty} c_n z^n \text{ diverges for } |z| > R \end{cases}$$

This number R, obviously unique, is called *the radius of convergence of the power series*, and is given by

$$R = \frac{1}{\limsup\limits_{n\to\infty} |c_n|^{1/n}} \tag{2}$$

Proof: We prove the theorem for R finite. The case $R = +\infty$ is left as an exercise.

• By definition of R, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \ \Rightarrow \ |c_n| < \left(\frac{1}{R} + \epsilon\right)^n$$

Let z such that $0 \leq |z| < R$. Then $|z|/R < 1$, so $\exists\ \epsilon$ such that

$$|z| \left(\frac{1}{R} + \epsilon\right) < 1$$

For this ϵ, there exists $M \in \mathbb{N}$ such that

$$n \geq M \quad \Rightarrow \quad \sum_{n=M}^{\infty} |c_n z^n| < \sum_{n=M}^{\infty} \left[\left(\frac{1}{R} + \epsilon\right)|z|\right]^n$$

The series is dominated by a convergence geometric series, so it converges.
• Conversely, if $|z| > R$, $\exists\ \epsilon$ such that

$$|z| \left(\frac{1}{R} - \epsilon\right) > 1$$

By definition of R, for this ϵ there exists a subsequence $(c_{n_k})_{k=1}^{\infty}$ for which

$$|c_{n_k}| > \left|\frac{1}{R} - \epsilon\right|^{n_k}$$

Then,

$$\forall k \in \mathbb{N}^*, |c_{n_k} z^{n_k}| > \left(|z| \left|\frac{1}{R} - \epsilon\right|\right)^{n_k}$$

Since the terms on the right-hand side of the inequality are unbounded, our series diverges. □

3.2 Power series and analyticity

Theorem: The power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic on the open disk $D_R(0)$ (i.e. the set $0 \leq |z| < R$) of convergence, and can be differentiated termwise in the disc of convergence:

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{(n-1)} \tag{3}$$

That series has the same radius of convergence as f.

Proof: • $\forall\ n \in \mathbb{N}^*$, $|nc_n|^{1/n} = |c_n|^{1/n} e^{\ln n/n}$, and $e^{\ln n/n} \to 1$ as $n \to \infty$, so the two series indeed have the same radius of convergence.

• Let $z_0 \in D_R(0)$ and $r \in \mathbb{R}$ such that $|z_0| < r < R$. $\forall\ n \in \mathbb{N}^*$, the function $u_n(z) = c_n(z^{n-1} + z_0 z^{n-2} + \ldots + z_0^{n-1})$ is continuous on $D_r(0)$ and on this disc satisfies

$$\sup_{z \in D_r(0)} |u_n(z)| \leq |c_n| n r^{n-1}$$

with $\sum_{n=1}^{\infty} n|c_n|r^{n-1}$ finite, as we proved in the point above

We conclude that $\sum_{n=1}^{\infty} u_n(z)$ converges uniformly on $D_r(0)$ and its sum is continuous on this disc, in particular in z_0.

Finally, we observe that $\forall z \in D_r(0) \setminus \{z_0\}$,

$$\sum_{n=1}^{\infty} u_n(z) = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0}$$

and that

$$\sum_{n=1}^{\infty} u_n(z_0) = \sum_{n=1}^{\infty} nc_n z_0^{n-1}$$

which completes our proof.

Corollary: Let $(c_n)_{n\in\mathbb{N}}$ a sequence of complex numbers and let $a \in \mathbb{C}$

• The power series centered in a, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ has derivatives of all orders on the open convergence disc $D_R(a)$, with derivatives given by

$$\forall\, p \in \mathbb{N}^* \ , \ f^{(p)}(z) = \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} c_n(z-a)^{n-p} \tag{4}$$

• In particular, we observe that

$$\forall\, p \in \mathbb{N} \ , \ c_p = \frac{f^{(p)}(a)}{p!} \tag{5}$$

In other words, a power series is the Taylor series of its sum.

The proof is a straightforward consequence of the previous theorem: it just requires a shift of $-a$, and a recursion to compute the expression for the successive derivatives

Important remarks:

• This corollary shows that if two power series centered at the same point a are equal in a neighborhood of a, they are equal term by term

• We will show in this course that every analytic function on an open set can be locally expanded in a power series. In other words, every analytic function on an open set has derivatives of all orders!
This is one of the most remarkable results of the theory of complex functions.

4 The exponential, trigonometric, and logarithmic functions

4.1 The exponential function

• The exponential function is defined, on a domain to be defined later, as the function which satisfies

$$\begin{cases} \dfrac{df}{dz} = f(z) \\ f(0) = 1 \end{cases} \tag{6}$$

Let us assume it has a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on some disc of convergence. Then

$$f'(z) = \sum_{n=1}^{\infty} nc_n z^{n-1} = f(z)$$

This implies that

$$\begin{cases} c_{n-1} = nc_n \ \ , \ \ n \geq 1 \\ c_0 = 1 \ \ \text{(initial condition)} \end{cases}$$

A rapid proof by induction shows that $c_n = 1/n!$. So if f exists and has a power series, this series is

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Since $\lim_{n\to\infty} (1/n!)^{1/n} = 0$, the series above is indeed convergent on all of $\mathbb{C}$, and

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{7}$$

is an entire function.

• Elementary properties:

Let a and b be two complex numbers

$$
\begin{aligned}
e^{a+b} &= \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{a^k b^{n-k}}{k!(n-k)!} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{n=k}^{\infty} \frac{b^{n-k}}{(n-k)!} = e^a e^b
\end{aligned}
$$

In particular, $\forall z \in \mathbb{C}$, $e^z e^{-z} = 1$ so $e^z \neq 0$

• Since the coefficients c_n of the power series for e^z are all real, $\overline{e^z} = e^{\overline{z}}$
Thus, for $(x, y) \in \mathbb{R}^2$,

$$
|e^{x+iy}| = \sqrt{(e^{x+iy}\overline{e^{x+iy}})} = \sqrt{e^{2x}} = e^x
$$

4.2 Hyperbolic functions

cosh and sinh are defined by

$$
\cosh z = \frac{e^z + e^{-z}}{2} \quad , \quad \sinh z = \frac{e^z - e^{-z}}{2} \tag{8}
$$

from which it is immediately clear that they are entire functions, with power series

$$
\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad , \quad \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}
$$

From the definition (8), it is straightforward to prove that $\forall (z, a, b) \in \mathbb{C}^3$

$$
\begin{aligned}
&\cosh^2 z - \sinh^2 = 1 \\
&\frac{d}{dz}(\sinh z) = \cosh z \\
&\frac{d}{dz}(\cosh z) = \sinh z \\
&\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b \\
&\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b
\end{aligned}
$$

4.3 Trigonometric functions

cos and sin are defined on $\mathbb{C}$ by

$$
\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad , \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{9}
$$

which means that cos and sin are entire, with power series

$$
\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad , \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}
$$

From the definition (9), one immediately gets the well-known Euler formula:

$$
\forall z \in \mathbb{C} \ , \ e^{iz} = \cos z + i \sin z \tag{10}
$$

Using (9), one can also rapidly rederive all the well-known formulae for the derivatives of cos and sin, $\cos(a+b)$, $\sin(a+b)$....

4.4 The logarithm function

• The logarithm function ln is defined such that $z = \ln w$ is a root of the equation $e^z = w$. Since $\forall z \in \mathbb{C}$, $e^z \neq 0$, the number 0 does not have a logarithm.

• Now, for $w \neq 0$, we may write $z = x + iy$ and

$$e^{x+iy} = w \quad \Leftrightarrow \quad \begin{cases} e^x = |w| \\ e^{iy} = \dfrac{w}{|w|} \end{cases}$$

- The first equation in the system has a unique solution, since exp is a bijection from $\mathbb{R}$ to $\mathbb{R}_+^*$:

$$x = \ln |w|$$

- Let $s = w/|w|$. $|s| = 1$ so $s = e^{iy}$ has a unique solution $y_0 \in [0, 2\pi)$, and infinitely many solutions $y = y_0 + 2\pi k$, $k \in \mathbb{Z}$
We conclude that every nonzero complex number has infinitely many logarithms, which differ from each other by integer multiples of $2\pi i$

If we write $w = re^{i\theta}$, $x = \ln r = \ln |w|$, and $y = \theta = \arg w$. Thus, for $w \neq 0$, we may write

$$\ln w = \ln |w| + i \arg w$$

and at the risk of repeating ourselves, the logarithm function is not single valued, because arg is not single valued.

• One usually adopts the convention that if $w \in \mathbb{R}_+^*$, $\ln w$ is the bijective real logarithm.

If $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, we define

$$a^b := e^{b \ln a}$$

We see that a^b is unique if $a \in \mathbb{R}_+^*$ according to our convention regarding the logarithm of a strictly positive real number, but in general has multiple values, which differ by $\exp(2\pi i k b)$, $k \in \mathbb{Z}$.

a^b will be unique independently of a when $b \in \mathbb{Z}$, as it should for integer powers of a complex number.

• One often likes to make the complex logarithm function single-valued by defining it as follows:

$$\mathrm{Ln} \ : z \in \mathbb{C} \mapsto \mathrm{Ln} z = \ln |z| + i \mathrm{Arg} z \tag{11}$$

where Arg$z$ is the argument of z in the interval $(-\pi, \pi]$. Ln defined above is single valued, but the price to pay for this is that Ln is not continuous across the negative real axis. Ln is however analytic on $\mathbb{C} \setminus \mathbb{R}_-$.

Ln is called *the principal branch of the logarithm.* In general, a branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The negative real axis is called a *branch cut* of this function.

One can construct infinitely many branches of ln by restricting the argument θ to be in the range $\alpha < \theta \leq \alpha + 2\pi$ with $\alpha \in [-\pi, \pi)$. The branch cuts are then the rays $\theta = \alpha$ including the origin. The origin is a point that is common to all the branch cuts, and is therefore called a *branch point* of the logarithm.

• Observe that $\ln e^z = z + i2\pi k$, $k \in \mathbb{Z}$
Even if ln is restricted to its principal branch Ln,

$$\mathrm{Ln} e^z \neq z \ \text{ unless } \ \Im(z) \in (-\pi, \pi]$$

Likewise, all we can say for an addition theorem is

$$\exists k \in \mathbb{Z} \text{ such that } \ln(z_1 z_2) = \ln z_1 + \ln z_2 + i2\pi k$$

Example:

$$\begin{aligned}
&\text{Ln}(-2i) = \ln 2 - i\frac{\pi}{2} \\
&\text{Ln}(-2) + \text{Ln}i = \ln 2 + i\pi + i\frac{\pi}{2} = \ln 2 + i\frac{3\pi}{2} \neq \text{Ln}(-2i)
\end{aligned}$$

We close this lecture by saying that the inverse trigonometric functions arccos and arcsin can also be defined in terms of the logarithm, through the exponential definitions of cos and sin seen previously.