# 1 Construction

## 1.1 Integrating a complex function over a curve in $\mathbb{C}$

• A natural way to construct the integral of a complex function over a curve in the complex plane is to link it to line integrals in $\mathbb{R}^2$ as already seen in vector calculus.

• We may understand this in two steps:

A) Consider a complex function $f(t) = u(t) + iv(t)$, for $t \in [a, b] \subset \mathbb{R}$, and $u$ and $v$ real valued functions. If $f$ is a continuous function, we may define

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt \tag{1}$$

This definition, combined with the elementary properties of addition and multiplication in $\mathbb{C}$ we saw in Lecture 1, means that the integral has many intuitive properties that are reminiscent of the properties of integrals of real functions. Let us mention a few without proof, as these proofs are elementary:

- Let $c \in [a, b]$ and $f$ continuous on $[a, b]$

$$\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt$$

$$\forall\lambda \in \mathbb{C}\,, \quad \int_a^b \lambda f(t)dt = \lambda\int_a^b f(t)dt$$

$$\Re\left(\int_a^b f(t)dt\right) = \int_a^b \Re(f(t))dt \;\;,\;\; \Im\left(\int_a^b f(t)dt\right) = \int_a^b \Im(f(t))dt$$

- Although the following property is also intuitive, let us prove that:

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt \tag{2}$$

If $\int_a^b f(t)dt = 0$, the inequality is trivial.
For $\int_a^b f(t)dt \neq 0$, let $\theta = \arg\left(\int_a^b f(t)dt\right)$

$$\left|\int_a^b f(t)dt\right| = \Re\left(e^{-i\theta}\int_a^b f(t)dt\right) = \Re\left(\int_a^b e^{-i\theta}f(t)dt\right) = \int_a^b \Re(e^{-i\theta}f(t))dt \leq \int_a^b |f(t)|dt \quad \square$$

With this preliminary step in place, we are ready to define integration on a general curve in $\mathbb{C}$.

B) Let $\gamma$ be a piecewise differentiable arc in the complex plane, with parametric equation

$$\gamma: \quad z = z(t)\;,\;\; a < t < b$$

If the function $f$ is continuous on $\gamma$, then $f(z(t))$ is continuous on $(a, b)$, and we define the integral of $f$ on $\gamma$ as the line integral

$$\int_\gamma f(z)dz := \int_a^b f(z(t))\frac{dz}{dt}dt \tag{3}$$

where the integral $\int_a^b$ may have to be split to match the intervals in which $z$ is differentiable.

The definition above only makes sense if the integral is independent of the way the arc $\gamma$ is parameterized. This is simple to check, using the rules for the change of variables for integrals of real valued functions. Imagine that another parameterization for $\gamma$ is given by

$$\gamma \;:\;\; \tau \in (\alpha, \beta) \;\mapsto\; z(t(\tau))$$

with $t\ :\tau\in(\alpha,\beta)\ \mapsto t(\tau)\in(a,b)$ piecewise differentiable. Then,

$$\int_\gamma f(z)dz=\int_a^b f(z(t))\frac{dz}{dt}dt=\int_\alpha^\beta f(z(t(\tau)))\frac{dz}{dt}\frac{dt}{d\tau}d\tau$$
$$=\int_\alpha^\beta f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau\qquad\square$$

## 1.2 Elementary properties

• Let $\gamma\ :\ z=z(t)\ ,\ t\in(a,b)$. We define the opposite arc, written $-\gamma$, by

$$-\gamma\ :\ z=z(-t)\ ,\ t\in(-b,-a)$$

Then,

$$\int_{-\gamma} f(z)dz=\int_{-b}^{-a} f(z(-t))\frac{d}{dt}\left[z(-t)\right]dt=-\int_{-b}^{-a} f(z(-t))\frac{dz}{dt}(-t)dt=-\int_a^b f(z(t))\frac{dz}{dt}(t)dt$$

where the last equality is obtained with a simple change of variable. Hence

$$\int_{-\gamma} f(z)dz=-\int_\gamma f(z)dz \tag{4}$$

• Linearity as an operator on functions

Let $f$ and $g$ be two continuous functions on the piecewise differentiable arc $\gamma$, and $(\alpha,\beta)\in\mathbb{C}^2$

$$\int_\gamma(\alpha f+\beta g)dz=\alpha\int_\gamma fdz+\beta\int_\gamma gdz \tag{5}$$

• Linearity as an operator on curves

Consider an arc $\gamma$ which can be subdivided into two piecewise-differentiable arcs $\gamma_1$ and $\gamma_2$, and $f$ a continuous function on $\gamma$. Then

$$\int_\gamma fdz=\int_{\gamma_1} fdz+\int_{\gamma_2} fdz=\int_{\gamma_2} fdz+\int_{\gamma_1} fdz \tag{6}$$

We can use this property to show that an integral over a closed curve does not depend on the starting point on the curve. Indeed, consider two such points P and Q, corresponding to different parameterizations, as shown in Figure 1. If we call $\gamma_1$ the part of $\gamma$ from $P$ to $Q$, and $\gamma_2$ the part of $\gamma$ from $Q$ to $P$,

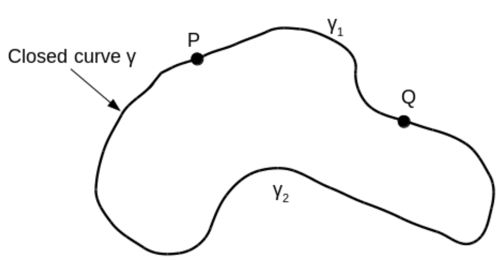


Figure 1: Closed curve $\gamma$ subdivided into the arcs $\gamma_1$ and $\gamma_2$

$$\int_\gamma fdz=\int_{\gamma_1} fdz+\int_{\gamma_2} fdz=\int_{\gamma_2} fdz+\int_{\gamma_1} fdz$$

The expression in the middle corresponds to the evaluation of the integral starting from the point $P$, while the expression on the right corresponds to the evaluation of the integral starting from the point $Q$.

### 1.3 An important example

We conclude this section with a very simple example which will play a fundamental role in the rest of this course.

Let $a \in \mathbb{C}$, and consider the integral

$$\int_\gamma \frac{dz}{z-a}$$

where $\gamma$ is the closed circle with radius $R$ and centered in $a$. A simple parameterization for $\gamma$ is $\gamma: \ \theta \in [0, 2\pi) \ \mapsto \ z(\theta) = Re^{i\theta} + a$. Thus

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} i d\theta = 2\pi i$$

## 2 The fundamental theorem of calculus for integrals in $\mathbb{C}$

### 2.1 Line integrals with respect to $x$ and $y$

The line integral with respect to $\overline{z}$ is defined as

$$\int_\gamma f(z) d\overline{z} := \overline{\int_\gamma \overline{f(z)} dz} \tag{7}$$

Line integrals with respect to $x = \Re(z)$ and $y = \Im(z)$ along the arc $\gamma$ are then naturally constructed as

$$\int_\gamma f(z)dx = \frac{1}{2}\left(\int_\gamma f(z)dz + \int_\gamma f(z)d\overline{z}\right) \quad , \quad \int_\gamma f(z)dy = \frac{1}{2i}\left(\int_\gamma f(z)dz - \int_\gamma f(z)d\overline{z}\right) \tag{8}$$

If we then write $f(z) = u(x, y) + iv(x, y)$, with $z = x + iy$, we have

$$\int_\gamma f(z)dz = \int_\gamma f(z)dx + i\int_\gamma f(z)dy = \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx) \tag{9}$$

which can be viewed as another definition for $\int_\gamma f(z)dz$, involving only line integrals of scalar functions, as already introduced in vector calculus.

### 2.2 Independence of path

We have just reduced the complex integral $\int_\gamma f(z)dz$ to line integrals of the form $\int_\gamma P(x, y)dx + Q(x, y)dy$. We will now recall a well-known result of vector calculus on independence of path to determine when $\int_\gamma f(z)dz$ only depends on the endpoints of $\gamma$ and not the actual path $\gamma$ describes.

**Theorem**: Let $\Omega$ be an open connected set of $\mathbb{R}^2$, and $P$ and $Q$ two functions that are continuous on $\Omega$, and potentially complex valued. The integral $\int_\gamma Pdx + Qdy$ depends only on the end points of $\gamma$ iff there exists a function $U(x, y)$ on $\Omega$ with the partial derivatives $P(x, y) = \frac{\partial U}{\partial x}$ , $Q(x, y) = \frac{\partial U}{\partial y}$.

*Proof*: The sufficient condition is straightforward: if such a $U$ exists, then

$$\int_\gamma Pdx + Qdy = \int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}\right) = \int_a^b \frac{d}{dt}\Big[U(x(t), y(t))\Big]dt = U(x(b), y(b)) - U(x(a), y(a))$$

for any arc $\gamma$ between the points $(x(a), y(a))$ and $(x(b), y(b))$.

- Conversely, if $\int_\gamma P(x, y)dx + Q(x, y)dy$ only depends on the end points, we can construct a single valued function $U$ by fixing a point $(x_0, y_0) \in \Omega$, and defining

$$U(x, y) = \int_\gamma P(x, y)dx + Q(x, y)dy$$

where $\gamma$ is any arc between $(x_0, y_0)$ and $(x, y)$. We now show that $U$ satisfies the conditions of the theorem.

Consider the point $(x + \Delta x, y)$, and any arc $\gamma'$ between $(x_0, y_0)$ and $(x + \Delta x, y)$. For $\Delta x$ sufficiently small, there exists an arc $\gamma''$ in $\Omega$ between $(x, y)$ and $(x + \Delta x, y)$, and parallel to the $x$-axis. By the independence of path of the integral, we can write

$$\begin{aligned} U(x + \Delta x, y) &= \int_{\gamma'} P(x,y)dx + Q(x,y)dy \\ &= \int_{\gamma} P(x,y)dx + Q(x,y)dy + \int_{\gamma''} P(x,y)dx \\ &= U(x,y) + \int_{\gamma''} P(x,y)dx \end{aligned}$$

Constructing arcs $\gamma''$ in this manner for all small $\Delta x$, we may write

$$\lim_{\Delta x \to 0} \frac{U(x + \Delta x, y) - U(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} P(x,y)dx = P(x,y)$$

where the last equation follows from the continuity of $P$. We thus have $\frac{\partial U}{\partial x} = P(x,y)$.
With a very similar proof, we would show that $\frac{\partial U}{\partial y} = Q(x,y)$, which concludes our proof. $\square$

## 2.3 The fundamental theorem of calculus for integrals in $\mathbb{C}$

Consider $f(z) = u(x,y) + iv(x,y)$, $P(x,y) = u(x,y) + iv(x,y)$, and $Q(x,y) = i(u(x,y) + iv(x,y))$. Then

$$\int_\gamma f(z)dz = \int_\gamma P(x,y)dx + Q(x,y)dy$$

The integral on the right-hand side depends on the end points if and only if there exists $F(x,y)$ such that $P(x,y) = \frac{\partial F}{\partial x}$, and $Q(x,y) = \frac{\partial F}{\partial y}$. If such an $F$ exists, then

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

Writing $F(z) = U(x,y) + iV(x,y)$, the equality above becomes Cauchy-Riemann equations for $U$ and $V$. So $F$ is analytic, with derivative $f$. We have proven the following theorem.

**Theorem (Fundamental theorem of calculus for integrals in $\mathbb{C}$)**: The integral $\int_\gamma f(z)dz$, with $f$ continuous on an open connected set $\Omega$ containing $\gamma$, depends only on the end points of $\gamma$ iff $f$ is the derivative of an analytic function $F$ in $\Omega$.
We say that $F$ is a primitive of $f$.

**Corollary**: if $f(z) = \frac{dF}{dz}$ where $F$ is analytic on an open connected set $\Omega$ and if $\gamma$ is a closed curve in $\Omega$, then

$$\oint_\gamma f(z)dz = 0 \tag{10}$$

Conversely, if $f$ is a continuous function on an open connected set $\Omega$ and is such that $\oint_\gamma f(z)dz = 0$ for any closed contour in $\Omega$, then $f$ has a primitive.

The proof of the latter is left to the reader as an enlightening exercise, close to what we have done previously in this lecture.

*Example*: • Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$

$$(z-a)^n = \frac{d}{dz}\left[\frac{(z-a)^{n+1}}{n+1}\right]$$

and $(z-a)^{n+1}/(n+1)$ is entire, so $\int_\gamma (z-a)^n dz = 0$ for all closed curves $\gamma$ in $\mathbb{C}$.

• For $n = -1$, we have already seen that the result does not hold

• For $n = -k$, $k \in \mathbb{N}^* \setminus \{1\}$, the result holds for any curve $\gamma$ in $\mathbb{C}$ that does not go through $a$

## 3 Integration with respect to arc length

We will often encounter integrals with respect to arc length, defined by

$$\int_\gamma f(z)ds := \int_\gamma f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt \tag{11}$$

As before, this only makes sense if the integral is independent of the parameterization. This can be verified easily, as well as the fact that

$$\int_{-\gamma} f(z)ds = \int_\gamma f(z)ds$$

The length of a curve $\gamma$ in the complex plane is given by

$$L(\gamma) = \int_\gamma ds = \int_\gamma |dz|$$

Finally, using Eq.(2) we have the triangle inequality:

$$|\int_\gamma f(z)dz| \leq \int_\gamma |f(z)||dz| \leq L(\gamma) \sup_{z\in\gamma} |f(z)| \tag{12}$$