A central result of complex analysis and of this lecture is the following theorem:

Theorem: Let f be an analytic function in the open connected set Δ' obtained by omitting a finite number of points ζ_i from an open disk Δ. If f satisfies the condition $\lim_{z\to\zeta_i}(z-\zeta_i)f(z)=0$ for all i, then $\int_\gamma f(z)dz=0$ for any closed rectifiable arc γ in Δ'.

We will prove this theorem, called *Cauchy's theorem*, step by step, starting with a very simple domain. Let us first begin with a short discussion of rectifiable arcs.

1 Rectifiable arcs

Consider the arc $\gamma : z = z(t)$, $a \le t \le b$. We have seen a way to define its length if it is piecewise differentiable. A more general definition is given by the least upper bound of all sums of the form

$$|z(t_1)-z(t_0)|+|z(t_2)-z(t_1)|+\ldots+|z(t_n)-z(t_{n-1})|$$

with $a=t_0<t_1<t_2<\ldots<t_{n-1}<t_n=b$.
If this least upper bound is finite, γ is said to be a rectifiable arc. Any piecewise differentiable arc is rectifiable, and in that case the two definitions of length are equivalent.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_\gamma f(z)dz = \lim_{\substack{N\to\infty,\\ |z(t_k)-z(t_{k-1})|\to 0}} \sum_{k=1}^{N} f(z(t_k))(z(t_k)-z(t_{k-1}))$$

In this course, we will never have to consider arcs which are not piecewise differentiable, but it is important to know that many of the theorems hold with weaker assumptions on γ.

2 Cauchy's Theorem for a rectangle

Theorem: Let R be the rectangle in the complex plane given by $a\le x\le b$, $c\le y\le d$, with $x=\Re(z)$ and $y=\Im(z)$, and ∂R is boundary curve, i.e. the arc following the boundary of R in the counterclockwise direction.
If a function f is analytic on an open set which contains R, then $\int_{\partial R} f(z)dz=0$.

The proof we give here is at once elegant and simple. It was first found by the French mathematician Edouard Goursat. The proof starts by bisecting R into four congruent rectangles R_1, R_2, R_3, and R_4, as shown in Figure 1, and looking for an upper bound for $\int_{\partial R} f(z)dz$ in terms of an integral on one of the smaller rectangles.

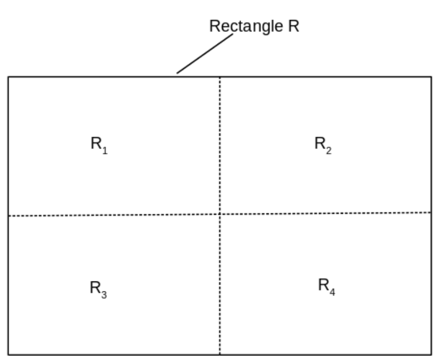

Figure 1: Bisection of the rectangle R into 4 congruent rectangles

For any rectangle $\tilde{R}$ inside R, R included, we define the number

$$\eta(\tilde{R}) = \int_{\partial\tilde{R}} f(z)dz$$

We can write

$$\eta(R) = \eta(R_1) + \eta(R_2) + \eta(R_3) + \eta(R_4)$$

since the integrals along shared sides cancel. Hence $\exists i$ such that $|\eta(R)| \leq 4|\eta(R_i)|$. The key idea is to repeat the bisection process in that R_i, to bound $\eta(R_i)$. After n subdivisions of this type, we can write

$$|\eta(R^{(n)})| \geq \frac{1}{4}|\eta(R^{(n-1)})| \geq \ldots \geq \frac{1}{4^n}|\eta(R)|$$

where $\eta(R^{(j)})$ is the rectangle used for the upper bound at the j^{th} subdivision.

Now, $\forall \delta > 0$, $\exists N \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ such that

$$\forall n \geq N \ , \ R^{(n)} \subset \{|z - z_0| < \delta \ , \ z \in \mathbb{C}\} \tag{1}$$

Furthermore, since f is analytic in R, $\forall \epsilon > 0$, $\exists \delta$ such that

$$|z - z_0| < \delta \ \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

For this δ, it is possible to subdivide R enough times such that $R^{(n)}$ satisfies Eq. (1). We can then write, using results from Lecture 4,

$$\eta(R^n) = \int_{\partial R^{(n)}} f(z)dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]dz$$

so that

$$|\eta(R^n)| \leq \epsilon \int_{\partial R^{(n)}} |z - z_0||dz|$$

Now, for z on $\partial R^{(n)}$, $|z - z_0| \leq d_n$, where d_n is the length of the diagonal of $R^{(n)}$. Thus, if L_n is the length of its perimeter,

$$|\eta(R^n)| \leq \epsilon d_n L_n$$

Finally, if d is the length of the diagonals of R and L the length of its perimeter,

$$d_n = \frac{d}{2^n} \quad , \quad L_n = \frac{L}{2^n}$$

We conclude that

$$|\eta(R)| \leq 4^n |\eta(R^{(n)})| \leq \epsilon d L$$

Since ϵ is arbitrarily small, $\eta(R) = 0$ $\square$

Our goal is to now generalize this result for cases in which f may not be analytic at a finite number of points ϵ_i inside R:

Theorem: Let f be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_i. If $\forall i \ , \ \lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0$, then

$$\int_{\partial R} f(z)dz = 0$$

Proof: Without loss of generality, we assume that f is not analytic at only one point ζ in R. We then subdivide R as shown in Figure 2, where S_0 is a square with center ζ.
Using the previous theorem,

$$\int_{\partial R} f(z)dz = \int_{\partial S_0} f(z)dz$$

Now, $\forall \epsilon > 0$, we may choose S_0 small enough that

$$\forall z \in \partial S_0 \ , \quad |f(z)| \leq \frac{\epsilon}{|z - \zeta|}$$

Hence,

$$\left| \int_{\partial R} f(z)dz \right| \leq \epsilon \int_{\partial S_0} \frac{|dz|}{|z - \zeta|} \leq \epsilon \frac{4l}{\frac{l}{2}} = 8\epsilon$$

where l is the length of a side of the square. Since ϵ is arbitrarily small, $\int_{\partial R} f(z)dz = 0$ $\square$

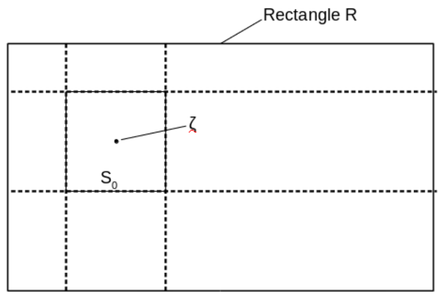

Figure 2: Subdivision of the rectangle R around the point ζ where f is not analytic

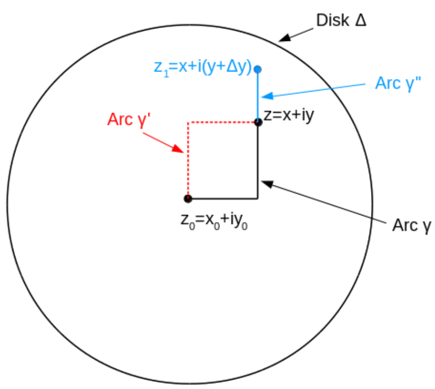

Figure 3: Disk Δ and relevant arcs for the first part of the proof of Cauchy's theorem

3 Cauchy's theorem for a disk

Theorem: If f is analytic in an open disk Δ, then $\int_\gamma f(z)dz = 0$ for every closed curve γ in Δ.

Proof: The proof is very similar in spirit to our proof for the independence of path in the previous lecture, but also uses Cauchy's theorem for a rectangle.

Consider the disk Δ centered in $z_0 = x_0 + iy_0$, and the point $z = x + iy$, and γ the arc that is horizontal from (x_0, y_0) to (x, y_0), and vertical from (x, y_0) to (x, y), as shown in Figure 3.

We define

$$F(z) = \int_\gamma f(z)dz$$

We have

$$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma''} f(z)dz = if(z)$$

where γ'' is the vertical line from (x, y) to $(x, y + \Delta y)$ (see Figure 3).

Now, by Cauchy's theorem on rectangles, one can also write

$$F(z) = \int_{\gamma'} f(z)dz$$

where γ' is shown in Figure 3. Thus, applying the same reasoning, we can also find

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta y} = f(z)$$

We conclude that $\partial F/\partial x = -i\partial F/\partial y$. We can thus say as in Lecture 4 that F is analytic, so that by the fundamental theorem of calculus

$$\int_\gamma f(z)dz = 0$$

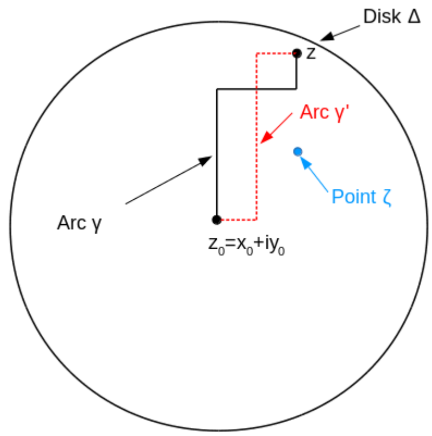

Figure 4: Disk Δ and relevant arcs for the second part of the proof of Cauchy's theorem

for any closed curve γ in Δ $\square$

We are now ready to prove Cauchy's theorem in its full extent, as stated at the beginning of these notes.

Proof: Without loss of generality, we can again assume that there is only one special point ζ in Δ. We define $F(z) = \int_\gamma f(z)dz$ in a similar way as before; we just have to be careful with the location of ζ with respect to the arcs we used in the proof.

First case: ζ lies neither on the line $x = x_0$ nor on the line $y = y_0$, where $z_0 = x_0 + iy_0$ is the center of Δ and the initial point of γ. Then it is possible to construct a path γ from z_0 to any $z \neq \zeta$ made only of horizontal and vertical line segments (three segments may be needed) with the last segment a vertical segment and where γ does not go through ζ. This can be seen in Figure 4.
It is then easy to show, in the same way as before, that $F_y(z) = if(z)$.

We know by Cauchy's theorem on a rectangle that $F(z) = \int_{\gamma'} f(z)dz$, with γ' shown in Figure 4, and that therefore $F_x(z) = f(z)$. We conclude that F is analytic, so $\int_\gamma f(z)dz = 0$ for any closed curve in Δ'.

Second case: ζ lines on the line $x = x_0$ or on the line $y = y_0$. In that case, one just moves the starting point z_0 for the definition of F away from $x_0 + iy_0$ to return to the first case $\square$