1 Cauchy's Integral Formula

1.1 Index of a point with respect to a closed curve

Let $z \in \mathbb{C}$, and a piecewise differentiable closed curve γ which does not pass through z. The value of the integral

$$\int_\gamma \frac{d\zeta}{\zeta - z}$$

is a multiple of $2\pi i$.

Indeed, let $\gamma : \zeta = \zeta(t)$, $a \leq t \leq b$, and consider the function

$$f(t) = \int_a^t \frac{1}{\zeta(u) - z} \frac{d\zeta}{du} du$$

Since γ does not pass through z, f is defined and continuous on $[a, b]$. Furthermore, for all t such that $\frac{d\zeta}{dt}(t)$ is continuous, we can write

$$f'(t) = \frac{1}{\zeta(t) - z} \frac{d\zeta}{dt} \qquad \Leftrightarrow \qquad \frac{d}{dt}\left[e^{-f(t)}(\zeta(t) - z)\right] = 0$$

Let us call $g(t) := e^{-f(t)}(\zeta(t) - z)$. Since γ is piecewise differentiable and since g is continuous on γ, we have

$$g(t) = Cst = g(a)$$

from which we conclude that

$$e^{f(t)} = \frac{\zeta(t) - z}{\zeta(a) - z}$$

For a closed curve γ, $\zeta(b) = \zeta(a)$, so

$$e^{f(b)} = e^{f(a)} = 1 \qquad \Leftrightarrow \quad \exists k \in \mathbb{Z} \;\; s.t. \;\; f(b) = 2\pi k i$$

Definition: The *index of the point z with respect to the closed curve γ* is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \tag{1}$$

n can be viewed as a quantity measuring the number of times a closed curve winds around a fixed point not on it. For this reason, n is often called the *winding number*.

Theorem: Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each open connected set of $\mathbb{C} \setminus \{\gamma\}$, and zero if this set is unbounded.

Proof: The function

$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

is integer valued on any open connected set of $\mathbb{C} \setminus \{\gamma\}$, and continuous on these sets. Since the image $f(\Omega)$ of any such set Ω is also connected, and the only connected subsets of the integers contain at most one point, f is constant.

In addition, for $|z|$ sufficiently large, there is a disk of radius R such that γ is contained in the disk but z is not. Then a direct application of Cauchy's theorem tells us that $n(\gamma, z) = 0$. This result then holds for the entire region by continuity.

1.2 Cauchy's integral formula

Theorem: Suppose that f is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point z not on γ

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \tag{2}$$

where $n(\gamma, z)$ is the index of z with respect to γ.

Proof: Let Δ be an open disk, γ a closed curve in Δ, and $z \in \Delta$ which does not lie on γ. We consider the function

$$F : \zeta \in \Delta \setminus \{z\} \mapsto \frac{f(\zeta) - f(z)}{\zeta - z}$$

From the hypotheses of the theorem, we know that F is analytic on $\Delta \setminus \{z\}$, and that

$$\lim_{\zeta \to z} F(\zeta)(\zeta - z) = 0$$

Hence, by Cauchy's theorem we know that $\int_\gamma F(\zeta)d\zeta = 0$, i.e

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = f(z)\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i n(\gamma, z)f(z) \quad \Box$$

Note that the properties of the function in the theorem can be relaxed to a function which is analytic in Δ except at a finite number of points ξ_i, provided that $\forall i, \lim_{z \to \xi_i}(z - \xi_i)f(z) = 0$. Cauchy's integral formula still holds in that case. The proof is left for the reader.

Cauchy's formula gives an expression for $f(z)$ only knowing that f is analytic in Δ and knowing the values of f on γ. This will be useful to prove many key theorems, and to study the local properties of functions. Here is a direct illustration:

Theorem (The mean value property for analytic functions): The value of an analytic function f at z is equal to the average of its values around any circle $|\zeta - z| = R$ inside the domain where it is analytic.

Proof: The result comes directly from Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(z + Re^{i\theta})d\theta$$

You probably came across a similar theorem for harmonic functions of real variables. The connection is clear, through the Cauchy-Riemann equations.

1.3 Derivatives of f

It is tempting to differentiate Cauchy's formula under the integral sign to obtain analogous formulae for the derivatives of f. To do so, we need a short lemma regarding that operation:

Lemma: Consider an open connected set Ω of $\mathbb{C}$, and γ an arc in Ω. If φ is continunous on γ, then

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}d\zeta$$

is analytic in $\Omega \setminus \{\gamma\}$, and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Proof: We prove this lemma by induction.

- The lemma is true for $n = 0$.

- Let us assume that it holds for $n - 1$: F_{n-1} is analytic on $\Omega \setminus \{\gamma\}$ for any φ continuous on γ, and $F_{n-1}'(z) = (n-1)F_n(z)\ \forall z \in \Omega \setminus \{\gamma\}$

• Let $z_0 \in \Omega \setminus \{\gamma\}$, and consider a neighborhood $D_\delta(z_0)$ that does not meet γ, and inside that neighborhood a smaller neighborhood $D_{\delta/2}(z_0)$. Observe that

$$z \in D_{\delta/2}(z_0) \;\Rightarrow\; \begin{cases} |z - z_0| < \frac{\delta}{2} \\ |\zeta - z| > \frac{\delta}{2}\,,\ \forall \zeta \in \gamma \end{cases}$$

For any continuous function φ on γ, we may write

$$\begin{aligned} F_n(z) - F_n(z_0) &= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta = \int_\gamma \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta \\ &= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta + (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta \end{aligned}$$

Let us define $\psi(\zeta) := \varphi(\zeta)/(\zeta - z_0)$, which is continuous on γ. We can rewrite the equality above as

$$F_n(z) - F_n(z_0) = \left[\int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^{n-1}} - \int_\gamma \frac{\psi(\zeta)}{(\zeta - z_0)^{n-1}} \right] + (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta \tag{3}$$

Now, $\forall z \in D_{\delta/2}(z_0)$,

$$\left| (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta \right| \le |z - z_0| \left(\frac{2}{\delta} \right)^n \int_\gamma |\psi(\zeta)||d\zeta|$$

so

$$\lim_{z \to z_0} (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta = 0$$

since ψ is continuous on γ and γ is rectifiable. Furthermore, we know by the induction hypothesis that the term in brackets in Eq. (3) goes to zero as $z \to z_0$. Hence, for any φ continuous on γ, F_n is continuous in z_0.

Defining

$$G_n(z) := \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n} d\zeta$$

we may write

$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + G_n(z)$$

By the induction hypothesis, the first term on the right goes to $G'_{n-1}(z_0) = (n-1)G_n(z_0)$ as $z \to z_0$, and from our previous point we also know that G_n is continuous, so we find

$$\lim_{z \to z_0} \frac{F_n(z) - F_n(z_0)}{z - z_0} = (n-1)G_n(z_0) + G_n(z_0) = nG_n(z_0) = nF_{n+1}(z_0) \quad \square$$

The lemma gives us the following important result:

Let f be a function which is analytic in an open connected set Ω. For any point z_0 in Ω, we consider a neighborhood $D_\delta(z_0) \subset \Omega$, and a circle C with center z_0 inside $D_\delta(z_0)$. For all points in the interior of C, we can use Cauchy's integral formula to write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Applying the lemma, we can say that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \tag{4}$$

is analytic in the interior of C. More generally,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{5}$$

is analytic in the interior of C. We have therefore proven the following central result of complex analysis:

An analytic function on the open connected set Ω has derivatives of all orders in Ω, which are themselves analytic.

2 Consequences of Cauchy's integral formula

2.1 Morera's theorem

Theorem: If f is defined and continuous in an open connected set Ω and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in Ω, then f is analytic in Ω.

Proof: From Lecture 4, we know that given the hypotheses of the theorem, f has a primitive in Ω. By the result we just found, f, the derivative of an analytic function in Ω, is analytic itself.

2.2 Cauchy's estimate

Suppose f is analytic in a disk $|z - z_0| \leq R$, and bounded on the circle γ given by $|z - z_0| = R$: $\forall\ z \in \gamma\ ,\ |f(z)| \leq M$ with $M \in \mathbb{R}_+$. Then

$$\begin{aligned} |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \int_\gamma \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| |d\zeta| \\ &\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R \end{aligned}$$

So we conclude that

$$|f^{(n)}(z_0)| \leq n! \frac{M}{R^n} \tag{6}$$

This inequality is known as Cauchy's estimate. It can be used for the well-known Liouville theorem below.

2.3 Liouville's theorem

Theorem: A bounded entire function is constant.

Proof: Let M be this bound. Then, using Cauchy's estimate, we have that

$$\forall\ z \in \mathbb{C}\ ,\ ,\forall R > 0, |f'(z)| \leq \frac{M}{R}$$

Hence $f'(z) = 0$, which means that f is constant.

2.4 The fundamental theorem of algebra

Theorem: Every polynomial of degree $n \geq 1$ has n roots.

Proof: Assume that $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ does *not* have a root. Then $g(z) := 1/P(z)$ is an entire function. Furthermore, g is bounded since

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|z|^n} = |a_n| \quad \Rightarrow \quad \lim_{|z|\to\infty} \frac{1}{P(z)} = 0$$

By Liouville's theorem, $1/P(z)$ must be a constant equal to zero, which is not possible. Hence, P has at least one root α, and we can write

$$P(z) = (z - \alpha)Q(z)$$

Repeating the steps for Q, we find that P must eventually have n roots. $\square$

2.5 Power series

Theorem: If f is analytic in an open connected set Ω which contains a closed disk $\overline{D_R(z_0)}$, then f has a power series expansion at z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

which is convergent for all $z \in D_R(z_0)$, with

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof: $\forall\ z \in D_R(z_0)$, $\forall \zeta \in C_R(z_0)$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} (\zeta - z_0)^{-n-1} (z - z_0)^n$$

Since convergence is uniform in $\zeta \in C_R(z_0)$, we can use Cauchy's formula to write

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \oint_{C_R(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta) \sum_{n=0}^{\infty} (\zeta - z_0)^{-n-1} (z - z_0)^n d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta) (\zeta - z_0)^{-n-1} d\zeta \right) (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \square \end{aligned}$$