1 Removable singularities

1.1 Riemann's removable singularity theorem

We have said that Cauchy's integral formula applied to functions which were not defined at a finite number of points in Δ, as long as $\lim_{z \to \xi_i} (z - \xi_i) f(z) = 0$ at these points ξ_i. We will now see that Cauchy's integral formula provides a natural way to extend such f to an analytic function on the entire set Δ. In other words, the ξ_i are *removable* singularities.

Theorem: Suppose that f is analytic in the open connected set Ω' obtained by omitting the point ξ from an open connected set Ω. There exists an analytic function in Ω which coincides with f in Ω' iff $\lim_{z \to \xi} (z-\xi) f(z) = 0$. The extended function is uniquely determined.

Proof: If the extended function exists, it is continuous in ξ, which guarantees uniqueness.

Likewise, by continuity of the extended function $\tilde{f}$, $\lim_{z \to \xi} (z - \xi) f(z) = \lim_{z \to \xi} (z - \xi) \tilde{f}(z) = 0$, which takes care of the necessary condition in the theorem.

For the sufficient condition, consider a circle centered at ξ and such that the circle C and the disk Δ corresponding to its interior are contained in Ω. For $z \neq \xi$ in Δ, we construct

$$F(\zeta) := \frac{f(\zeta) - f(z)}{\zeta - z}$$

F has two singularities in Δ: $\zeta = z$ and $\zeta = \xi$. We have

$$\lim_{\zeta \to z} (\zeta - z) F(\zeta) = 0$$

by continuity of f in z. We also have

$$\lim_{\zeta \to \xi} (\zeta - \xi) F(\zeta) = 0$$

by the hypothesis of the theorem. Therefore, applying Cauchy's theorem to F,

$$\int_C F(\zeta) d\zeta = 0$$

Hence, for any $z \neq \xi$ in Δ,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1}$$

Now, we know from Lecture 6 that the right-hand side of (1) is an analytic function of z throughout the inside of C. It is therefore continuous in ξ, with value

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \xi} d\zeta$$

In other words,

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \ , \quad \forall z \in \Omega \tag{2}$$

is the desired analytic extension of f in the whole open connected set Ω.

1.2 Taylor's theorem

Let us apply the previous result to the function

$$F(z) := \frac{f(z) - f(\xi)}{z - \xi}$$

where $(z, \xi) \in \Omega^2$, with $z \neq \xi$, Ω is an open connected set, as before, and f is analytic on Ω.
Observe that

$$\lim_{z \to \xi} (z - \xi) F(z) = 0 \quad , \quad \lim_{z \to \xi} F(z) = f'(\xi)$$

By the previous theorem, there exists an analytic function f_1 on Ω such that

$$\begin{cases} f_1(z) = F(z) & \text{if } z \neq \xi \\ f_1(\xi) = f'(\xi) \end{cases}$$

$\forall\, z \in \Omega$, we may thus write

$$f(z) = f(\xi) + (z - \xi) f_1(z)$$

This expansion for f can also be applied to f_1: there exists an analytic function f_2 on Ω such that

$$f_1(z) = f_1(\xi) + (z - \xi) f_2(z)$$

with

$$\begin{cases} f_2(z) = \frac{f_1(z) - f_1(\xi)}{z - \xi} & \text{if } z \neq \xi \\ f_2(\xi) = f_1'(\xi) \end{cases}$$

Continuing the recursion, we can write the general form

$$f_{n-1}(z) = f_{n-1}(\xi) + (z - \xi) f_n(z)$$

In this process, we obtained the following expansion for f:

$$f(z) = f(\xi) + (z - \xi) f_1(\xi) + (z - \xi)^2 f_2(\xi) + \ldots + (z - \xi)^{n-1} f_{n-1}(\xi) + (z - \xi)^n f_n(z)$$

Furthermore, by direct differentiation at $z = \xi$, we have

$$f^{(n)}(\xi) = n! f_n(\xi)$$

We have just prove Taylor's theorem, stated below:

Theorem: If f is analytic in an open connected set Ω containing ξ, it is possible to write

$$f(z) = f(\xi) + f'(\xi)(z - \xi) + \frac{f''(\xi)}{2}(z - \xi)^2 + \ldots + \frac{f^{(n-1)}(\xi)}{(n-1)!}(z - \xi)^{n-1} + (z - \xi)^n f_n(z) \qquad (3)$$

where f_n is analytic in Ω.

Note that Taylor's formula, given by Eq.(3), is *not* a Taylor series, unlike the power series we have seen at the end of the previous lecture. It is very useful nonetheless, especially because there is a simple expression for f_n in terms of f:

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - \xi)^n (\zeta - z)} d\zeta \qquad (4)$$

To see why (4) holds, we start with Cauchy's integral formula,

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)}{(\zeta - z)} d\zeta$$

and we represent f_n in the integrand using Taylor's formula. When we do so, the first term is

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - \xi)^n (\zeta - z)} d\zeta$$

The other terms all have the following form, to within a constant factor:

$$g_k(\xi) = \int_C \frac{d\zeta}{(\zeta - \xi)^k (\zeta - z)} \quad 1 \leq k \leq n - 1$$

Observe first that one may write

$$g_k(\xi) = \int_C \frac{\varphi(\zeta)}{(\zeta - \xi)^k} d\zeta \quad 1 \leq k \leq n - 1$$

with φ continuous on C. Hence, from the lemma proved in Lecture 6, $\forall\, k \in [\![2, n-1]\!]$, $g'_k(\xi) = k g_{k+1}(\xi)$. So all we need to do is evaluate

$$g_1(\xi) = \int_C \frac{d\zeta}{(\zeta-\xi)(\zeta-z)} = \frac{1}{z-\xi}\int_C \left(\frac{1}{\zeta-z} - \frac{1}{\zeta-\xi}\right) d\zeta = \frac{1}{z-\xi}(2\pi i - 2\pi i) = 0$$

where we recognized the definition of the winding number to derive the third equality above. We conclude that $g_1(\xi) = 0$, and thus $g_k(\xi) = 0$ for $k \in [\![2, n-1]\!]$, from which the formula (4) for f_n follows.

Corollary: If f is analytic in the open connected set Ω and if there exists $\xi \in \Omega$ such that $f^{(n)}(\xi) = 0$ for all $n \in \mathbb{N}$, then $f \equiv 0$ in Ω.

Proof: Let us first prove that this is true for a disk $D_R(\xi) \subset \Omega$ with boundary C.
By Taylor's formula, if the hypotheses of the theorem hold, $\forall n \in \mathbb{N}$, we can write

$$f(z) = (z-\xi)^n f_n(z) \quad \text{,with } \; f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\xi)^n(\zeta-z)} d\zeta$$

Now, let us call M the maximum value of $|f(z)|$ on C. $\forall\, z \in D_R(\xi)$, we can write

$$|f_n(z)| \le \frac{1}{2\pi}\frac{M}{R^n}\frac{2\pi R}{R - |z-\xi|}$$

Hence, $\forall\, z \in D_R(\xi)$

$$|f(z)| \le \left|\frac{z-\xi}{R}\right|^n \frac{MR}{R-|z-\xi|}$$

And since this is true for all $n \in \mathbb{N}$, $f(z) = 0$ in $D_R(\xi)$.

Note that there is another, more direct way to this part of the proof: just consider the coefficients of the power series of f in the neighborhood of ξ, as seen in Lecture 6! Here we propose a slightly different proof for this first part as a way to practice with Taylor's formula and upper bounds.

To complete the proof, we now have to extend the result from $D_R(\xi)$ to Ω. For this purpose, consider the following two sets:

$$E_1 := \{z \in \Omega \mid f^{(n)}(z) = 0\; \forall n \in \mathbb{N}\} \qquad , \qquad E_2 := \{z \in \Omega \mid \exists n \in \mathbb{N} : f^{(n)}(z) \neq 0\}$$

E_1 and E_2 are such that $E_1 \cap E_2 = \{\emptyset\}$. The first part of the proof shows that E_1 is an open set. Furthermore, by continuity of f and all its derivatives, E_2 is open as well. Now, since Ω is connected and $\Omega = E_1 \cup E_2$, either $E_1 = \{\emptyset\}$ or $E_2 = \{\emptyset\}$. According to the hypotheses of the theorem, $E_1 \neq \{\emptyset\}$. Therefore $E_2 = \{\emptyset\}$, and $f \equiv 0$, as desired $\square$

2 Zeros and poles

2.1 Zeros of a function

Let f be an analytic function in Ω which is not identically zero, and $\xi \in \Omega$. From what we have just seen, there exists a first integer N such that $f^{(N)}(\xi) \neq 0$. Then, by Taylor's formula, we can write

$$f(z) = (z-\xi)^N f_N(z)$$

with f_N analytic and such that $f_N(\xi) \neq 0$. We say that ξ is a *zero of order* N of f.

Observe that f_N is continuous, so $\exists\, \delta > 0$ such that $\forall\, z$ such that $0 < |z-\xi| < \delta$, $f(z) \neq 0$: the zeros of f are isolated. This can be reformulated with the following theorem:

Identity Theorem: If f and g are analytic in Ω, and if $f = g$ on a set which has an accumulation point in Ω, then $\forall\, z \in \Omega$, $f(z) = g(z)$.
The theorem is immediate by looking at the Taylor formula for $f - g$, as long as we remember what an accumulation point is:

- A point z of a subset S is called an isolated point of S if there exists a neighborhood of z whose intersection with S reduces to the point z
- An accumulation point is a point of $\overline{S}$ which is not an isolated point.

A trivial yet important consequence of the identity theorem is as follows:
If f is analytic in Ω and identically zero in a nonempty connected open subset of Ω, then $f \equiv 0$ in Ω.
Likewise, if f is identically zero on an arc in Ω which does not reduce to a point, $f \equiv 0$ in Ω.

2.2 Poles of a function

Consider a function f which is analytic in a neighborhood of ξ, but perhaps not in ξ itself. ξ is then called an *isolated singularity*.

If $\lim_{z\to\xi} f(z) = \infty$, ξ is said to be a *pole of f*.
By continuity, there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(\xi)$ with $z \neq \xi$. Thus, $g(z) := 1/f(z)$ is analytic for all z such that $0 < |z-\xi| < \delta$. Furthermore, g can be analytically extended on $D_\delta(\xi)$, with $g(\xi) = 0$ since $\lim_{z\to\xi}(z-\xi)g(z) = 0$.
The order of the pole of f in ξ is the order N of the zero of g in ξ. We can write

$$f(z) = \frac{f_N(z)}{(z-\xi)^N} \quad , \quad 0 < |z-\xi| < \delta$$

with f_N analytic and nonzero in a neighborhood of ξ.

Definition: A function which is analytic in an open connected set Ω except for isolated poles is called a *meromorphic function*.

If f has a pole of order N at ξ, then we can use Taylor's formula to write:

$$(z-\xi)^N f(z) = a_N + a_{N-1}(z-\xi) + \ldots + a_1(z-\xi)^{N-1} + \varphi(z)(z-\xi)^N$$

with φ analytic at $z = \xi$. Hence, for $z \neq \xi$, we may write

$$f(z) = \frac{a_N}{(z-\xi)^N} + \frac{a_{N-1}}{(z-\xi)^{N-1}} + \ldots + \frac{a_1}{z-\xi} + \varphi(z)$$

where the sum of the terms in blue is called *the singular part of f at ξ*.

2.3 Essential singularity

Let f be analytic in a disk $0 < |z-\xi| < \delta$ with the center ξ removed.

(i) If $\lim_{z\to\xi} f(z)$ exists or if $\lim_{z\to\xi}(z-\xi)f(z) = 0$, then ξ is a removable singularity, and f extends to an analytic function on the whole disk $|z-\xi| < \delta$

(ii) If $\lim_{z\to\xi} f(z) = \infty$, ξ is said to be a pole. In this case, $f(z) = (z-\xi)^{-N} f_N(z)$ with $N \in \mathbb{N}^*$ the order of the pole, f_N analytic in a neighborhood of ξ, and $f_N(\xi) \neq 0$.

(iii) If neither (i) nor (ii) holds, ξ is said to be an *essential singularity*.

Example: $f(z) = \exp(1/z)$ has an essential singularity at $\xi = 0$.

The behavior of a function near an essential singularity is quite extreme, as illustrated by the following theorem.

Casorati-Weierstrass theorem: An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.

Proof: Suppose the statement is false: $\exists\, z_0 \in \mathbb{C}$ and $\delta > 0$ and $\epsilon > 0$ such that

$$|f(z) - z_0| > \epsilon \quad \text{for all } z \text{ such that } \quad |z-\xi| < \delta$$

Thus,

$$\lim_{z\to\xi}\frac{f(z)-z_0}{z-\xi}=\infty$$

so that the function

$$g(z):=\frac{f(z)-z_0}{z-\xi}$$

has a pole at $z=\xi$. We may then write $g(z)=(z-\xi)^{-N}g_N(z)$ with $N\in\mathbb{N}^*$ and g_n analytic in a neighborhood of ξ. In other words,

$$f(z)=(z-\xi)^{1-N}g_N(z)+z_0$$

If $N=1$, f has a removable singularity at $z=\xi$.
If $N>1$, $f-z_0$ has a pole at $z=\xi$, and so does f.
Both possibilities are excluded by the hypothesis of the theorem, so the statement must be true. $\square$