Lecture 8 – Local properties of analytic functions – Part 2 MATH-GA 2451.001 Complex Variables

1 The argument principle (Part 1) – Counting the zeros of an analytic function

Theorem: Consider a function f which is analytic in a disk D and which does not vanish identically. Let ζ_j be the zeros of f, each zero being counted as many times as its order indicates. For every closed curve γ in D which does not pass through a zero, we have

$$\sum_j n(\gamma, \zeta_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \tag{1}$$

Proof: Using Taylor's formula, we may write, for $z \in D$

$$f(z) = (z-\zeta_1)(z-\zeta_2)\dots(z-\zeta_n)g(z)$$

with g analytic and such that $g(z) \neq 0$, $\forall z \in D$.
Hence, for any $z \in D$ such that $z \neq \zeta_j$,

$$\frac{f'(z)}{f(z)} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2} + \dots + \frac{1}{z-\zeta_n} + \frac{g'(z)}{g(z)}$$

By Cauchy's theorem,

$$\int_\gamma \frac{g'(z)}{g(z)} = 0$$

so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = \sum_{j=1}^{n} n(\gamma, \zeta_j) \tag{2}$$

We have just proved the result for the situation for which f has a finite number of zeros. For the case in which f has infinitely many zeros in D, the proof can be extended as follows.

Let us say that f has infinitely many zeros in D. Since γ is inside D, it is contained in a disk D' smaller than D. Now, since f is not identically zero, it can only have finitely many zeros inside D'. This result follows from a combination of the Bolzano-Weierstrass theorem and the identity theorem. Thus, the formula (2) holds inside D'. It holds inside D as well since for the zeros of ζ_j of f outside of D', $n(\gamma, \zeta_j) = 0$. This concludes our proof □

Observe that the integral on the right of (1) can be represented as

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{f'(z)}dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}dt = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}dt = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w}$$

$f\circ\gamma$ is a closed curved Γ. The equality (1) in the theorem can thus be interpreted as the equality

$$n(\Gamma, 0) = \sum_j n(\gamma, \zeta_j)$$

The name "argument principle" can be given the following intuitive – although not at all rigorous – interpretation:

$$\text{“}\ \frac{dw}{w} = d(\ln w) = d(\ln|w| + i\arg w)\ \text{”}$$

Note the quotes around these equalities, which should be seen as formal equalities and nothing else. For any curve that does not pass through 0, $\ln|w|$ is well defined, so by the fundamental theorem of calculus the contribution of the real part in the formal equalities above to the integral is zero when one integrates over a closed curve.

Let me stress, once more, that this is just intended to provide an intuition, but it is absolutely not rigorous.

The most useful application of the theorem is for the case when γ is a circle (or more generally a simple closed curve) so that

$$n(\gamma, \zeta_j) = \begin{cases} 0 \text{ if } \zeta_j \text{ is outside } \gamma \\ 1 \text{ if } \zeta_j \text{ is inside } \gamma \end{cases}$$

The formula in the theorem then gives a formula for the number of zeros enclosed by γ. This formula is at the heart of a number of numerical methods to locate the zeros of an analytic function.

Finally, we will soon see the true, full version of the argument principle, involving the counting of the poles as well, once we learn the calculus of residues.

2 Open mapping theorem

Let $a \in \mathbb{C}$. Applying the argument principle theorem to $f(z) = a$, the roots $\zeta_j(a)$ of the equation $f(z) = a$ satisfy

$$\sum_j n(\gamma, \zeta_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} dz$$

for any closed curve γ which does not pass through a point z such that $f(z) = a$.

Now, if two points a and b are both in the interior of $\Gamma = f \circ \gamma$, or both in its exterior,

$$n(\Gamma, a) = n(\Gamma, b)$$
$$\Leftrightarrow \sum_j n(\gamma, \zeta_j(a)) = \sum_j n(\gamma, \zeta_j(b))$$

For the special case in which γ is a circle (or a simple closed curve), that means that f takes the values a and b an equal number of times inside γ. This leads to the following theorem.

Theorem: Suppose that f is analytic at z_0, and suppose that $f(z) - w_0$ has a zero of order N at $z = z_0$. Then for any $\epsilon > 0$ sufficiently small there exists $\delta > 0$ such that $\forall a \in \mathbb{C}$ such that $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly N roots in the disk $|z - z_0| < \epsilon$.

Proof: Let $z_0 \in \mathbb{C}$, and choose $\epsilon > 0$ such that

- f is analytic in $|z - z_0| < \epsilon$
- z_0 is the only zero of $f(z) - w_0$ in this disk
- $f'(z) \neq 0$ for z such that $0 < |z - z_0| < \epsilon$

Consider the circle γ corresponding to the boundary of that disk: $|z - z_0| = \epsilon$.
$\Gamma = f \circ \gamma$ is a closed set. w_0 is in the complement of this closed set, so there exists $\delta > 0$ such that the disk $D_\delta(w_0)$ does not intersect Γ.
By what we have seen at the beginning of this section, f takes all values $a \in D_\delta(w_0)$ the same number of times inside γ.

Indeed, $n(\Gamma, a) = n(\Gamma, w_0)$ and $\sum_j n(\gamma, \zeta_j(w_0)) = n$ so $\sum_j n(\gamma, \zeta_j(a)) = n$. And since we have made sure, by choosing ϵ small enough, that $f'(z) \neq 0$ for $z \neq z_0$, and the $\zeta_j(a)$ are *simple* roots of $f(z) = a$ $\square$

Corollary 1: If f is analytic at z_0 and z_0 is a simple zero of $f(z) - w_0$, then there exists a neighborhood of z_0 and a corresponding neighborhood of w_0 on which f is one-to-one.

Corollary 2: **Open mapping theorem**

A nonconstant analytic function maps open sets to open sets.

Proof: The previous theorem implies that the image of every sufficiently small disk $|z - z_0| < \epsilon$ contains a neighborhood $|w - w_0| < \delta$.

3 The maximum principle

3.1 The maximum principle

Consider a function f which is analytic and nonconstant on an open connected set Ω. By the open mapping theorem, $\forall\ z_0 \in \Omega$, there exists an open disk $|w - f(z_0)| < \epsilon$ contained in the image of Ω. In this open disk, there exist points w such that $|w| > |f(z_0)|$. In other words, $|f|$ does not reach a maximum in z_0. This proves the following theorem:

Theorem: Maximum modulus principle

If f is analytic and nonconstant in an open connected set Ω, then its modulus $|f|$ has no maximum in Ω.

The theorem is often reformulated in the following equivalent way:

If f is defined and continuous on a closed bounded set E, and analytic in the interior of E, then the maximum of $|f|$ on E is assumed on the boundary of E.

Illustration: remember problem 2 in HW #2, stating that there does not exist analytic functions of $z = x+iy$ whose modulus is equal to $K/\cosh x$, with $K \neq 0$ constant.

The maximum modulus principle gives us a quick proof of this for functions which are analytic on an open connected set which contains a subset of the imaginary axis in its interior: $\forall y \in \mathbb{R},\ z = 0 + iy$ is a maximum of $|f|$, since $\cosh 0$ is a minimum of cosh. Since f cannot be constant by hypothesis, this would contradict the maximum modulus principle, so no such f exists.

3.2 The lemma of Schwarz

Theorem: If f is analytic in the disk $|z| < 1$ and satisfies the conditions

$$\begin{cases} f(0) = 0 \\ |f(z)| \leq 1\ ,\ \forall z \in D_1(0) \end{cases}$$

then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Furthermore, if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = Az$, with $A \in \mathbb{C}$ such that $|A| = 1$

Proof: For $z \in D_1(0)$, consider the function g defined by

$$g(z) = \begin{cases} \frac{f(z)}{z}\ ,\ z \neq 0 \\ f'(0)\ ,\ z = 0 \end{cases}$$

From the hypotheses of the theorem, we know that g is analytic. By the maximum modulus principle,

$$\forall R\ :\ 0 < R < 1\ ,\ |g(z)| \leq \frac{1}{R} \quad \text{for } z \in \overline{D}_R(0)$$

since $|g(z)| \leq 1/R$ on the circle $C_R(0)$. Letting $R \to 1$, we conclude that $|g(z)| \leq 1$ for $|z| < 1$. This concludes the proof of the first part of the theorem.

Regarding the second part of the theorem, if $|f(z)| = |z|$ holds for some $z \neq 0$ in $D_1(0)$, then g reaches its maximum in the disk, so g is constant. The same reasoning holds in $|f'(0)| = 1$ $\square$

The lemma of Schwarz is more powerful than one may at first think because its conditions can be generalized substantially:

Consider an analytic function f on the unit disk, which maps $D_1(0)$ onto itself, and $z_0 \in D_1(0)$, with $f(z_0) = w_0$. The Möbius transformation

$$T(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

maps the unit disk onto itself, and is bijective, as we have seen in Lecture 3. Likewise,

$$S(w) = \frac{w - w_0}{1 - \overline{w_0} w}$$

maps the unit disk onto itself. We conclude that the map $S \circ f \circ T^{-1}$ maps $D_1(0)$ onto itself, and $S(f(T^{-1}(0))) = 0$. We can apply the lemma of Schwarz to $S \circ f \circ T^{-1}$, to obtain the inequality

$$\left| \frac{f(T^{-1}(w)) - f(z_0)}{1 - \overline{f(z_0)} f(T^{-1}(w))} \right| \leq |w|$$

With $z = T^{-1}(w)$, this can be rewritten as

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| \quad , \forall (z, z_0) \in (D_1(0))^2$$

The lemma of Schwarz can be generalized further to functions with upper bound $M \in \mathbb{R}_+$ instead of 1: we then apply the lemma to $f(z)/M$, and M will appear on the right hand side of the inequality. Likewise, if f satisfies the conditions on a disk of radius R instead of the unit disk, we apply the theorem to $f(Rz)$, and $1/R$ will appear on the right-hand side of the inequality.