1 Chains and cycles

1.1 Chains

Let Ω be an open set in $\mathbb{C}$. A chain in Ω is a finite collection $\gamma_j \ : [a_j, b_j] \ \to \Omega \ j = 1, \ldots, N$ of piecewise continuously differentiable curves in Ω.
Writing $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ for a given chain, we can integrate a continuous function f in Ω along Γ as follows:

$$\int_\Gamma f(z)dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)dz$$

1.2 Cycles

A cycle in Ω is a chain $\Gamma = \sum_{j=1}^{N} \gamma_j$ where each point $z \in \mathbb{C}$ is an initial point of just as many of the γ_j as it is a terminal point. In other words, a cycle is a finite sum of closed curves.

As an illustration, the index of a point z with respect to the cycle Γ is

$$n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z} = \sum_{j=1}^{N} \int_{\gamma_n} \frac{d\zeta}{\zeta - z}$$

Observe that the integrals in the sum on the right-hand side of the last equality above may not be over closed curves.

2 Simple connectivity and homology

2.1 Simple connected sets in $\mathbb{C}$

Below, we start this section with an unusual definition for simple connectedness. Its weakness is that it is not general, in the sense that it cannot be used in $\mathbb{R}^n$ with $n \geq 3$. However, we will show later in this course that for $\mathbb{C}$, it is equivalent to the more common definition, which says that any simple closed curve can be shrunk to a point continuously in the set. And the advantage of our unusual definition is that it is more convenient for the proof of the general form of Cauchy's theorem.

Definition: An open connected set $\Omega \subset \mathbb{C}$ is said to be simply connected if its complement with respect to $\hat{\mathbb{C}}$ is connected.

Theorem: An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$

Proof: • Let us start with the necessary condition: for any cycle $\gamma \in \Omega$, the complement of Ω in $\hat{\mathbb{C}}$ must be in one of the regions determined by γ (interior or exterior), since this complement is connected. Since $\{\infty\}$ belongs to this complement, this must be the unbounded region defined by γ. From Lecture 6, we thus know that $n(\gamma, z) = 0 \ \forall z \notin \Omega$.

• We prove the sufficient condition by direct construction. Specifically, we will show that if a region Ω is not simply connected, then one can construct a cycle γ in Ω and find a point z_0 which does not belong to Ω such that $n(\gamma, z_0) \neq 0$.

Let us assume that the complement of Ω in $\hat{\mathbb{C}}$ is $A \cup B$, with A and B disjoint closed sets, with a shortest distance $\delta > 0$ between the two sets. Let us say that B is the unbounded set, so A is bounded. We conver A with a net of squares S whose sides have length $l < \delta/\sqrt{2}$, constructed in such a way that $z_0 \in A$ lies at the center of a square, as shown in Figure 1.

Consider the cycle $\gamma = \sum_j \partial S_j$, where ∂S_j is the boundary curve of each square S_j, and where the sum is taken over the net covering A.
Observe first that $n(\gamma, z_0) = 1$ since z_0 belongs to only one of the squares in the net.

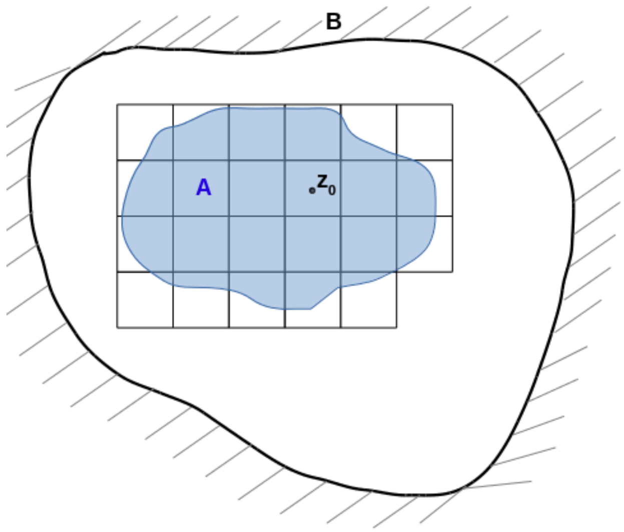

Figure 1: Net of squares covering the set A without intersecting the set B

Furthermore, it is clear that γ does not belong to B. Now, the key is to realize that γ does not belong to A either, in the sense that there exists a cycle $\tilde{\gamma}$ contained in Ω such that $n(\tilde{\gamma}, z_0) = n(\gamma, z_0) = 1$. Indeed, $\tilde{\gamma}$ is directly obtained from γ by observing that in the integral corresponding to $n(\gamma, z_0)$, all the sides of the squares contained in A are traversed exactly twice, in opposite directions, and therefore cancel. This concludes our proof $\square$

2.2 Homology

Definition: A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, z) = 0$ for all z in the complement of Ω in $\hat{\mathbb{C}}$.
One write $\gamma \sim 0 \pmod{\Omega}$, or often $\gamma \sim 0$ when it is clear that one is talking about Ω.
$\gamma_1 \sim \gamma_2$ means $\gamma_1 - \gamma_2 \sim 0$

Note that with this notation, the previous theorem can be written as
Theorem: An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $\gamma \sim 0$ for all γ in Ω.

3 The general form of Cauchy's theorem

We now have all the tools required to give Cauchy's theorem in its most general form.

Theorem: If f is analytic in the open set Ω, then $\int_\gamma f(z)dz = 0$ for every cycle γ which is homologous to zero in Ω.

Proof: Consider γ such that $\gamma \sim 0 (\text{mod } \Omega)$, and the set

$$E = \{z \in \mathbb{C} \setminus \gamma \ : \ n(\gamma, z) = 0\}$$

which is open.
We define the function

$$g : \ (z, \zeta) \in \Omega^2 \mapsto \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & , \ z \neq \zeta \\ f'(z) \ , \ z = \zeta \end{cases}$$

g is continuous in both its variables. Furthermore, $\forall \zeta_0 \in \Omega$, $\tilde{g} : \ z \mapsto g(z, \zeta_0)$ is analytic in Ω since $\lim\limits_{z \to \zeta_0} (z - \zeta_0) g(z, \zeta_0) = 0$, so ζ_0 is a removable singularity.

We now introduce the function h on $\mathbb{C}$ defined by

$$\begin{cases} h(z) = \frac{1}{2\pi i}\int_\gamma g(z,\zeta)d\zeta \ \text{ if } z \in \Omega \\ h(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \ \text{ if } z \in E \end{cases}$$

h is indeed defined on all of $\mathbb{C}$ since $\mathbb{C} \setminus \Omega \subset E$, so that $\Omega \cup E = \mathbb{C}$, and also because the two definitions of h agree on $\Omega \cap E$ since $n(\gamma, z) = 0 \, \forall z \in \Omega \cap E$.

We claim that h is entire. h is analytic in E by the same argument that was made in Lecture 6.
To prove that h is analytic on Ω, we consider the following lemma:

Suppose $[a, b] \subset \mathbb{R}$ and let φ be a continuous complex-valued function on $\Omega \times [a, b]$ such that $\forall t \in [a, b]$, $z \mapsto \varphi(z, t)$ is analytic on Ω. If we define F by

$$F(z) = \int_a^b \varphi(z,t)dt \ , \ \forall z \in \Omega$$

F is analytic on Ω.

The proof of that lemma is as follows. Let $z_0 \in \Omega$ and $R > 0$ such that $\overline{D}_R(z_0) \subset \Omega$.
$\forall z \in D_R(z_0)$,

$$\begin{aligned} F(z) = \int_a^b \varphi(z,t)dt &= \frac{1}{2\pi i}\int_a^b \left(\int_{|\zeta - z_0| = R} \frac{\varphi(\zeta,t)}{\zeta - z}d\zeta \right) dt \\ &= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R} \left(\int_a^b \varphi(\zeta,t)dt \right) \frac{d\zeta}{\zeta - z} \end{aligned}$$

where the interchange in the order of integration can be justified by parametrizing the integral over $|\zeta - z_0| = R$ and applying the result for interchanging the order of integration for continuous functions on rectangles. Now, $\int_a^b \varphi(\zeta, t)dt$ is a continuous function of ζ, so again by the same reasoning as in Lecture 6, F is analytic on $D_R(z_0)$.

We conclude that h is entire. Now, for $|z|$ sufficiently large, $n(\gamma, z) = 0$ so $z \in E$, and since f is bounded on γ, $h(z) \to 0$ as $|z| \to \infty$. We conclude that h is bounded, and $h \equiv 0$ by Liouville's theorem.

Hence $\forall z \in \Omega \setminus \gamma$,

$$\frac{1}{2\pi i}\int_\gamma g(z,\zeta)d\zeta = 0 \quad \Leftrightarrow \quad n(\gamma,z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

This is the generalized version of Cauchy's integral formula, which we can now use to prove Cauchy's theorem.

Let $z_0 \in \Omega \setminus \gamma$, and consider $F(z) = (z - z_0)f(z)$, $(z \in \Omega)$

$$\int_\gamma f(z)dz = \int_\gamma \frac{F(z)}{z - z_0}dz = 2\pi i n(\gamma, z_0)F(z_0) = 0$$

This completes this very elegant proof, first proposed by John Dixon in the *Proceedings of the American Mathematical Society*, Volume 29, Number 3, August 1971.

Corollary 1: If f is analytic in a simply connected open set Ω, then $\int_\gamma f(z)dz = 0$ for all cycles in Ω.

This follows directly from Cauchy's theorem, and the theorem in page 1 of these notes.

Corollary 2: If f is analytic and nonzero in a simply connected open region Ω, then it is possible to define single-valued analytic branches of $\ln[f(z)]$ and $\sqrt[n]{f(z)}$ in Ω.

Indeed, by Cauchy's theorem we know that

$$\int_\gamma \frac{f'(z)}{f(z)}dz = 0$$

for all cycles in Ω. We then know that there exists an analytic function F such that $F'(z) = f'(z)/f(z)$ $\forall z \in \Omega$.
In other words,

$$\frac{d}{dz}\left[f(z)e^{-F(z)}\right] = 0 \;\Leftrightarrow\; f(z) = Ae^{F(z)} \;,\; A \in \mathbb{C}^*$$

Now, choose $z_0 \in \Omega$ and one of the infinitely many values of $\ln[f(z_0)]$.

$$\exp\left[F(z) - F(z_0) + \ln[f(z_0)]\right] = \frac{f(z)}{A}e^{-F(z_0)}f(z_0) = f(z)$$

We can therefore define a single-valued, analytic branch of the logarithm of f as

$$\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$$

The definition of $\sqrt[n]{f}$ follows from this result, as $\forall z \in \Omega$ we write $\sqrt[n]{f} = \exp\left[\frac{1}{n}\ln(f(z))\right]$