MATH-GA.2702.001 Fluid Dynamics

Lecture 11 - Stokes flow

In the previous lecture, we considered some classical steady flows for which the nonlinear term in the Navier-Stokes equation was identically zero, or negligible, or was not needed to compute the flow.

In the present lecture, we will consider more general configurations, but still in the laminar flow regime, and where the nonlinear term is negligible. This will be achieved by considering the very small Reynolds number regime, $Re \ll 1$. The flows in this regime are usually called Stokes flows. They are applicable to situations in which the flow speed is very small, or the fluid has a high viscosity, or one is interested in very small scales.

I/ Stokes flow

1) Defining equations

We have seen that the flow of a viscous fluid with constant and uniform density and viscosity is given by:

$$\begin{cases} St \, \dfrac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = - \vec{\nabla} p + \dfrac{1}{Fr} \vec{e}_g + \dfrac{1}{Re} \vec{\nabla}^2 \vec{u} \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} Re\left(St\,\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{\nabla}p - \frac{1}{Fr}\vec{e_z}\right) = \vec{\nabla}^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

For Stokes flow, we consider the regime $St \sim 1$, $Fr \sim 1$, and $Re \longrightarrow 0$.
In that case, the momentum equation reduces to: $\vec{\nabla}^2\vec{u} = 0$.
Now, we can generalize this somewhat, by changing the way we scaled the ∇p term, so that $|Re\,\vec{\nabla}p|$ is also of order 1. We then have the general form of Stokes flow:

$$\boxed{\begin{cases} \vec{\nabla}^2\vec{u} = \vec{\nabla}p \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases} \qquad \text{Stokes Flow}}$$

In this limit, we thus have an elliptic partial differential equation with no time derivative: $\vec{u}$ responds instantaneously to the boundary conditions.

2) Uniqueness of Stokes flow

Let us demonstrate explicitly that the flow in the Stokes regime is uniquely determined by the boundary conditions.

Consider Stokes flow with no slip boundary conditions within a volume V bounded by a boundary ∂V with velocity $\vec{u}_B$. Assume there are two solutions $\vec{u}_1$ and $\vec{u}_2$ to the equations

$$\begin{cases} \vec{\nabla}^2 \vec{u} = \vec{\nabla} p \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

with the appropriate no slip boundary condition.

Then $\vec{v} = \vec{u}_2 - \vec{u}_1$ satisfies

$$\begin{cases} \vec{\nabla}^2 \vec{v} = \vec{\nabla} p \\ \vec{\nabla} \cdot \vec{v} = 0 \end{cases} \quad \text{and} \quad \vec{v} = \vec{0} \quad \text{on } \partial V$$

First note that $\vec{\nabla}^2 \vec{v} = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla} \times (\vec{\nabla} \times \vec{v})$ so an alternate form of the equation is

$$\begin{cases} \vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla} p \\ \vec{\nabla} \cdot \vec{v} = 0 \end{cases}$$

We thus have:

$$0 = \int_V \vec{v} \cdot [\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) + \vec{\nabla} p] \, dV$$

$$\underset{\underline{\underline{\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}}}}{=} \int_V \left\{ \vec{\nabla} \cdot \left[(\vec{\nabla} \times \vec{v}) \times \vec{v} \right] + |\vec{\nabla} \times \vec{v}|^2 + \vec{\nabla} \cdot (p\vec{v}) \right\} dV$$

$$= \int_V |\vec{\nabla} \times \vec{v}|^2 \, dV$$

Thus, $\vec{\nabla} \times \vec{v} = 0$ pointwise inside V. There exists a scalar function ϕ such that $\vec{v} = \vec{\nabla}\phi$, which satisfies

$$\vec{\nabla}^2 \phi = 0 .$$

The boundary condition $\vec{v} = \vec{0}$ on ∂V then imply $\phi =$ cst throughout V, i.e $\vec{v} = \vec{0}$ throughout V ▪

3) <u>Time reversibility in Stokes flow</u>

We have just showed that Stokes flow is uniquely determined by the velocity condition on the boundary of the domain. By linearity of the equation, if the velocity on the boundary is reversed, then so is the velocity everywhere in the fluid.
By undoing the boundary motion one first created, one can therefore undo the mixing occurring in Stokes flow, as we saw in the video demonstration in class. Furthermore, since the equations for Stokes flow do not explicitly depend on time, the rate at which one does or undoes the boundary motion does not matter for the reversibility property; all that matters is that one undoes the motion exactly in the reverse order as one does it.

II] Low Reynolds number flow around a sphere

1) Stokes flow around a sphere

We consider an axisymmetric flow at low Reynolds number around a sphere. In spherical coordinates (ρ, θ, φ), all the physical quantities will only depend on ρ and θ.

As we saw in Lecture 6, the incompressibility condition then allows us to express the flow $\vec{u}$ in terms of a stream function Ψ:

$$u_\rho = \frac{1}{\rho^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} \quad , \quad u_\theta = -\frac{1}{\rho \sin\theta}\frac{\partial \Psi}{\partial \rho}$$

As we have seen, the momentum equation can be written as:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = -\vec{\nabla}p$$

$$\Leftrightarrow \vec{\nabla}\times\vec{\omega} = -\vec{\nabla}p$$

Taking the curl of this equation,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\omega}) = \vec{0}$$

Now, $\vec{\omega} = \vec{\nabla}\times\vec{u} = -\dfrac{1}{\rho\sin\theta} L\Psi \, \vec{e}_\varphi$

with $L := \dfrac{\partial^2}{\partial\rho^2} + \dfrac{\sin\theta}{\rho^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$

$$\vec{\nabla}\times\vec{\omega} = -\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}L\Psi\right)\vec{e}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{\sin\theta}L\Psi\right)\vec{e}_\theta$$

$$= -\frac{1}{\rho\sin\theta}\left[\frac{1}{\rho}\frac{\partial}{\partial\theta}L\Psi\,\vec{e}_\rho - \frac{\partial}{\partial\rho}L\Psi\,\vec{e}_\theta\right]$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\omega}) = \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \rho} L \Psi \right) + \frac{1}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\rho \sin\theta} \frac{\partial}{\partial \theta} L \Psi \right) \right] \vec{e_\varphi}$$

$$= \frac{1}{\rho \sin\theta} \left[\frac{\partial^2}{\partial \rho^2} L \Psi + \frac{\sin\theta}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} L \Psi \right) \right] \vec{e_\varphi}$$

$$= \frac{1}{\rho \sin\theta} L^2 \Psi \vec{e_\varphi}$$

The momentum equation thus reduces to:

$$L^2 \Psi = 0$$

* If the sphere is immersed in the uniform flow $U \hat{z}$, the boundary condition at infinity is

$$u_\rho = U \cos\theta \qquad u_\theta = -U \sin\theta$$

$$\Rightarrow \quad \Psi = U \frac{\rho^2}{2} \sin^2\theta \quad \text{as} \quad \rho \longrightarrow \infty$$

* On the sphere, we impose no-slip boundary conditions:

$$\left. \frac{\partial \Psi}{\partial \theta} \right|_{(a,\theta)} = 0 \Rightarrow \Psi(a,\theta) = 0$$ (since Ψ is defined up to a free constant).

$$\left. \frac{\partial \Psi}{\partial \rho} \right|_{(a,\theta)} = 0$$

where a is the radius of the sphere.

As is customary in this situation, we look for a solution of the form

$$\Psi(\rho, \theta) = f(\rho)\, g(\theta)$$

Given the boundary condition at ∞, we set $g(\theta) = \sin^2\theta$ (and can return to this assumption if we cannot find a satisfactory solution of this form).

We then have $L^2 \Psi = 0$

$$\Leftrightarrow \left(\frac{d^2}{d\rho^2} - \frac{2}{\rho^2}\right)^2 f(\rho) = 0 \quad (*)$$

We seek solutions for f of the form $f(\rho) \propto \rho^n$. Plugging such a solution in $(*)$, we find the following condition on n:

$$n(n-1)(n-2)(n-3) - 2n(n-1) - 2(n-2)(n-3) + 4 = 0$$
$$\Leftrightarrow (n-4)(n-2)(n-1)(n+1) = 0$$

Thus, the most general solution is $f(\rho) = \frac{A}{\rho} + B\rho + C\rho^2 + D\rho^4$

The boundary condition at ∞ implies that $D = 0$, $C = \frac{U}{2}$

On the sphere, $\begin{cases} \frac{A}{a} + Ba + U\frac{a^2}{2} = 0 \\ -\frac{A}{a^2} + B + Ua = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{a}A + aB = -\frac{Ua^2}{2} \\ -\frac{1}{a^2}A + B = -Ua \end{cases}$

$$\Leftrightarrow \begin{cases} A = \frac{a^3}{4} U \\ B = -\frac{3}{4} a U \end{cases}$$

We therefore have:

$$\Psi(\rho, \theta) = \frac{U}{4}\left(\frac{a^3}{\rho} - 3a\rho + 2\rho^2\right)\sin^2\theta$$

so that

$$u_\rho = \frac{U}{2\rho^2}\left(\frac{a^3}{\rho} - 3a\rho + 2\rho^2\right)\cos\theta$$

$$u_\theta = \frac{U}{4\rho}\left(\frac{a^3}{\rho^2} + 3a - 4\rho\right)\sin\theta$$

$$\vec{\omega} = -\frac{3}{2}\frac{Ua}{\rho^2}\sin\theta\ \vec{e_\varphi}$$

$$\Rightarrow \vec{\nabla}\times\vec{\omega} = -\frac{3Ua}{\rho^3}\cos\theta\ \vec{e_\rho} - \frac{3}{2}\frac{Ua}{\rho^3}\sin\theta\ \vec{e_\theta} = -\frac{\partial p}{\partial\rho}\vec{e_\rho} - \frac{1}{\rho}\frac{\partial p}{\partial\theta}\vec{e_\theta}$$

$$\Rightarrow p = p_\infty - \frac{3}{2}\frac{Ua}{\rho^2}\cos\theta$$

where p_∞ is the fluid pressure at $\rho \to \infty$.

We see that with respect to the equator, which is the plane given by $\theta = \frac{\pi}{2}$, u_ρ is odd and u_θ is even: the streamlines are symmetric before and after the sphere, as one may expect intuitively.
The streamlines for this flow are plotted in the figure in the next page.

All these figures were obtained for $U = 1$, $a = 1$.

Fig. 1. Contours of the stream function for Stokes flow around the sphere in solid lines. Contours of the stream function for inviscid potential flow are shown in dashed lines for comparison.

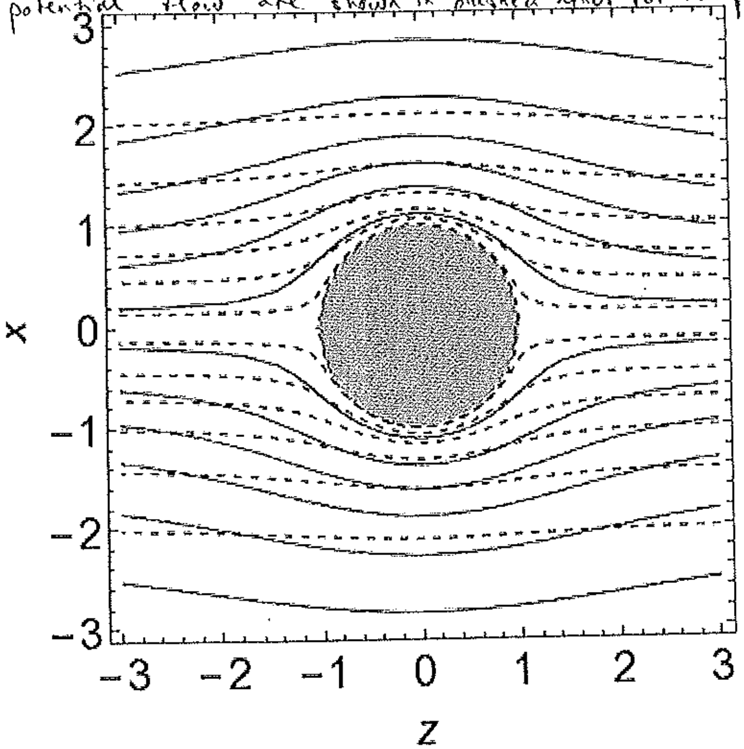


Fig. 2. Dark solid lines: contours of the stream function as in Figure 1. The grey lines and color code correspond to the contours of $\|u\|$.

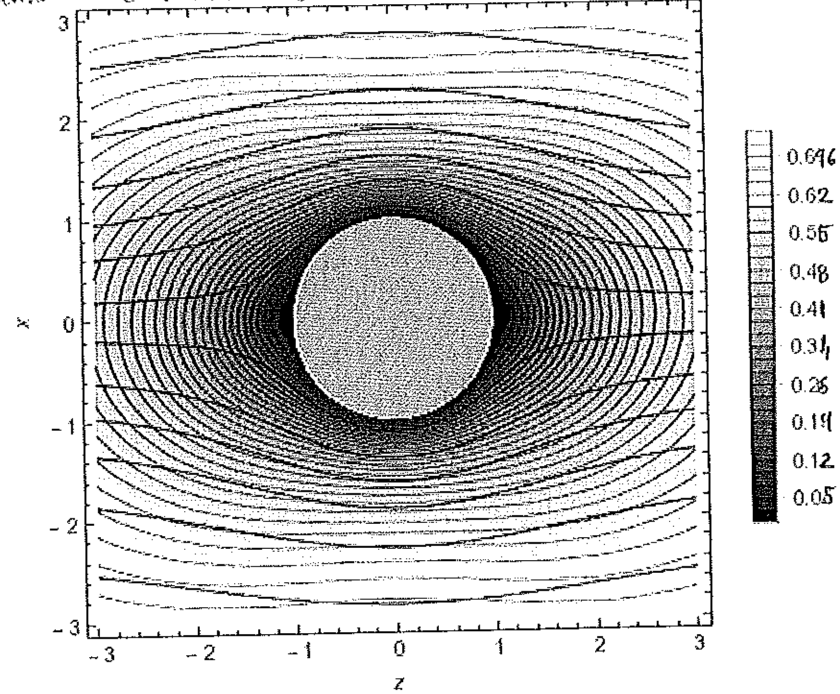


Fig. 3. Contours of $p - p_\infty$ for the flow in figures 1 and 2. The solid lines correspond to positive values, the dashed lines to negative values.

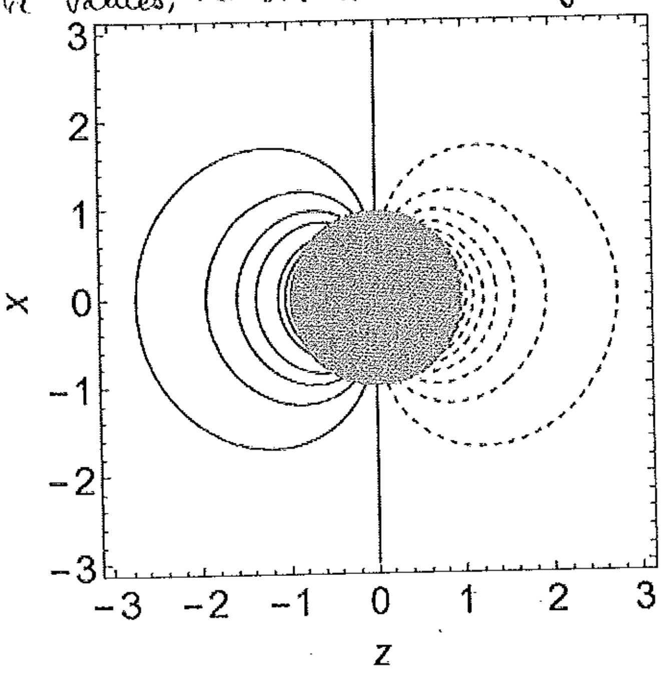
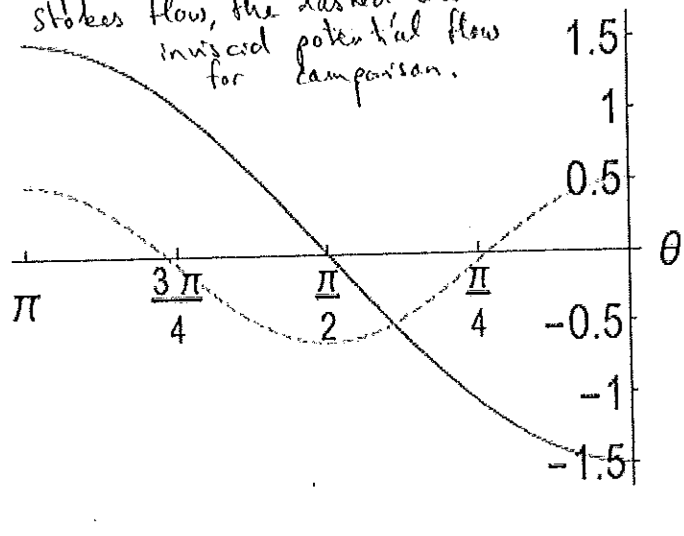


Fig. 4. $p - p_\infty$ as a function of the latitude θ, along the upper hemisphere of the sphere, for $y = 0$. $\theta = 0$ corresponds to downstream, $\theta = \pi$ to upstream. The solid line corresponds to Stokes flow, the dashed line to inviscid potential flow for comparison.



2) Drag on the sphere

To compute the drag of the fluid on the sphere, we need, given the symmetry of the problem, to consider the reduced stress tensor:

$$\overleftrightarrow{\Pi} = \begin{bmatrix} \Pi_{rr} & \Pi_{r\theta} \\ \Pi_{r\theta} & \Pi_{\theta\theta} \end{bmatrix}$$

where we have used the symmetry of $\overleftrightarrow{\Pi}$.
$\Pi_{\theta\theta}$ represents normal stresses on planes with normal vector $\vec{e}_\theta$. They are irrelevant for our spherical geometry.

In spherical coordinates, we have

$$\Pi_{rr} = -p + 2\frac{\partial u_\rho}{\partial \rho} \quad , \quad \Pi_{r\theta} = \rho \frac{\partial}{\partial \rho}\left(\frac{u_\theta}{\rho}\right) + \frac{1}{\rho}\frac{\partial u_\rho}{\partial \theta}$$

$$\frac{\partial u_\rho}{\partial \rho} = \frac{U}{2}\left(\frac{3a}{\rho^2} - \frac{3a^3}{\rho^4}\right)\cos\theta$$

On the sphere, $\frac{\partial u_\rho}{\partial \rho} = 0$.

$$\rho\frac{\partial}{\partial \rho}\left(\frac{u_\theta}{\rho}\right) = -\rho\frac{U}{4}\left(\frac{4a^3}{\rho^5} + \frac{6a}{\rho^3} - \frac{4}{\rho^2}\right)\sin\theta$$

$$\Rightarrow \quad \rho\frac{\partial}{\partial \rho}\left(\frac{u_\theta}{\rho}\right)\bigg|_{\rho=a} = -\frac{3}{2}\frac{U}{a}\sin\theta$$

$$\frac{1}{\rho}\frac{\partial u_\rho}{\partial \theta}\bigg|_{\rho=a} = 0$$

Finally, at $\rho = a$, $p = p_\infty - \frac{3}{2}\frac{U}{a}\cos\theta$

We are now ready to compute the net drag in the z-direction:

$$D_{z\text{-direction}} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{3}{2}\frac{U}{a}\cos\theta - p_\infty\right)\cos\theta \, a^2 \sin\theta \, d\theta \, d\varphi$$

$$- \int_0^{2\pi}\int_0^{\pi} \frac{3}{2}\frac{U}{a} \sin\theta \, \sin\theta \, a^2 \sin\theta \, d\theta \, d\varphi$$

$$= 3\pi U a \left[\int_0^{\pi} \overbrace{\left(\cos^2\theta \sin\theta - \cancel{p_\infty \cos\theta \sin\theta} \right)}^{\text{contribution to drag from pressure}} d\theta - \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{\text{contribution to drag from viscous stresses}} \right]$$

$$= 3\pi U a \left(\overset{\text{pressure}}{\frac{2}{3}} + \underset{\text{viscous}}{\frac{4}{3}} \right) = 6\pi U a$$

$$D_{z\text{-direction}} = 6\pi U a.$$

This calculation was done in dimensionless units, so the dynamic viscosity μ does not appear in the result. In dimensional units, one readily finds:

$$\boxed{D_{z\text{-direction}} = 6\pi \mu U a}$$

3) Fall velocity

As we discussed in class, the experimental physicist Robert Millikan used this formula, together with his PhD student Harvey Fletcher, to determine the charge of an electron. Let us compute the fall velocity of the drop through the fluid.
According to Newton's law, the drop will not accelerate if the drag is exactly balanced by the net sum of the gravity and buoyancy forces:

$$6\pi \mu U a = \frac{4\pi}{3} a^3 (\rho_{drop} - \rho_f) g$$

$$\Leftrightarrow U = \frac{2}{9\mu} a^2 (\rho_{drop} - \rho_f) g$$

The formula depends on the radius of the drop explicitly, as one would expect. This allowed Millikan and Fletcher to measure the radius of the oil drops, and therefore their (apparent, or buoyant) weight, which was subsequently needed to calculate the charge of the electron by balancing the electric force and gravitational force when a potential difference is applied in the experiment.

II | Stokes' flow past an infinite circular cylinder

1) Stokes' paradox

Consider a Stokes' flow past an infinitely long circular cylinder. Given the geometry of the problem, we work in polar coordinates (r, θ).
The flow is incompressible, so we introduce a stream function once more:

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \quad , \quad u_\theta = -\frac{\partial \Psi}{\partial r}$$

If at ∞ the flow is the uniform flow $U\hat{x}$, then the boundary condition on Ψ as $r \to +\infty$ is:

$$\Psi \longrightarrow U r \sin\theta \quad \text{as} \quad r \longrightarrow +\infty$$

At the cylinder, we have: $\left.\frac{\partial \Psi}{\partial \theta}\right|_{(a,\theta)} = 0 \Rightarrow \Psi(a,\theta) = 0$

$$\left.\frac{\partial \Psi}{\partial r}\right|_{(a,\theta)} = 0$$

As before, we seek a solution by separation of variables. The boundary condition at ∞ suggests:

$$\Psi(r,\theta) = f(r)\sin\theta$$

Thus, $u_r = \frac{f(r)}{r}\cos\theta \quad , \quad u_\theta = -f'(r)\sin\theta$

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \left[-\frac{1}{r}\frac{d}{dr}\left(r f'(r)\right)\sin\theta + \frac{f(r)}{r^2}\sin\theta\right]\vec{e}_z$$

$$= -\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) f\right]\sin\theta\,\vec{e}_z$$

$$= - L f(r) \sin\theta \, \vec{e}_z \quad \text{with} \quad L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$$

$$\vec{\nabla} \times \vec{\omega} = - \frac{1}{r} L f(r) \cos\theta \, \vec{e}_r + \frac{d}{dr}\Big(L f(r) \Big)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\omega} \right) = - \frac{1}{r^2} L f(r) \sin\theta \, \vec{e}_z + \frac{1}{r}\frac{d}{dr}\left[r \frac{d}{dr}\big(L f(r) \big) \right] \sin\theta \, \vec{e}_z$$

$$= L^2 f(r) \sin\theta \, \vec{e}_z .$$

In Stokes flow, the momentum equation implies $\vec{\nabla} \times (\vec{\nabla} \times \vec{\omega}) = \vec{0}$

$$\Longleftrightarrow \quad L^2 f(r) = 0$$

The most general solution to this equation is

$$f(r) = K_1 r^3 + K_2 r \ln r + K_3 r + \frac{K_4}{r}$$

Applying the boundary conditions at ∞, we have: $K_1 = K_2 = 0$

$f(a) = 0 \Longrightarrow K_3 a + \frac{K_4}{a} = 0 \Longleftrightarrow K_4 = - K_3 a^2$

$f'(a) = 0 \Longrightarrow K_3 - \frac{K_4}{a^2} = 0 \Longleftrightarrow K_4 = K_3 a^2$

Thus $K_3 = K_4 = 0$, so $f(r) = 0 \ \forall r \in \mathbb{R}_+$, which contradicts the boundary condition at ∞.

The conclusion is as follows: there is no satisfactory steady solution of the two-dimensional Stokes equations representing flow of an unbounded fluid past a circular cylinder.

Here, we only proved this result for solutions which can be written in separable form, but the result holds in general. This is known as Stokes' paradox.

2) Oseen's equations

The source of the paradox is that by working and the problem on a finite domain, as in the class textbook, one would find that the solution ψ behaves like $C r \ln r$ for large (and finite) r. The velocity thus behaves like $C \ln r$.

We then have $Re\,\vec{u}\cdot\vec{\nabla}\vec{u} \sim Re\,C^2 \frac{\ln r}{r}$

$$\vec{\nabla}^2\vec{u} \sim C\frac{\ln r}{r^2}$$

The two terms are comparable when

$$r\,Re\,C = O(1)$$

i.e $\quad r = O\left(\frac{1}{Re}\right)$

At such distances and beyond, the Stokes approximation is not justified, and inertial terms need to be included.

In order to include these effects while keeping the problem linear, Oseen developed an asymptotic theory which suggests the following approximation, for our problem of interest:

- At small r, $a < r < \frac{1}{Re}$, solve Stokes' equation
- At large r, $r > \frac{1}{Re}$, solve

$$\rho U \frac{\partial \vec{u}}{\partial x} + \vec{\nabla} p - \mu \vec{\nabla}^2 \vec{u} = 0 \quad , \quad \vec{\nabla} \cdot \vec{u} = 0$$

- Match the two solutions at $r = \frac{1}{Re}$ to determine the free constants.